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# Canonized Rewriting and Ground AC Completion Modulo Shostak Theories : Design and Implementation

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**Abstract** AC-completion efficiently handles equality modulo associative and commutative function symbols. When the input is ground, the procedure terminates and provides a decision algorithm for the word problem. In this paper, we present a modular extension of ground AC-completion for deciding formulas in the combination of the theory of equality with user-defined AC symbols, uninterpreted symbols and an arbitrary signature disjoint Shostak theory  $X$ . Our algorithm, called  $AC(X)$ , is obtained by augmenting in a modular way ground AC-completion with the canonizer and solver present for the theory  $X$ . This integration rests on canonized rewriting, a new relation reminiscent to normalized rewriting, which integrates canonizers in rewriting steps.  $AC(X)$  is proved sound, complete and terminating, and is implemented to extend the core of the ALT-ERGO theorem prover.

**Keywords** decision procedure · associativity and commutativity · rewriting · AC-completion · SMT solvers · Shostak’s algorithm

## 1 Introduction

The mechanization of mathematical proofs is a research domain that receives an increasing interest among mathematicians and computer scientists. In particular, automated theorem provers (ATP) are now used in several contexts (*e.g.* proof of programs, interactive provers) to prove “simple” but overwhelming intermediate results. While more and more efficient, ATP have difficulties to handle some mathematical operators, such as union and intersection of sets, which satisfy the following associativity and commutativity (AC) axioms

$$\begin{aligned}\forall x.\forall y.\forall z. u(x, u(y, z)) &= u(u(x, y), z) && (A) \\ \forall x.\forall y. u(x, y) &= u(y, x) && (C)\end{aligned}$$

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Indeed, the mere addition of AC axioms to a prover will usually glut it with plenty of useless equalities which will strongly impact its performances<sup>1</sup>. In order to avoid this drawback, built-in procedures have been designed to efficiently handle AC symbols. For instance, SMT-solvers incorporate dedicated decision procedures for some *specific* AC symbols such as arithmetic or boolean operators. On the contrary, algorithms found in resolution-based provers such as AC-completion allow a powerful *generic* treatment of user-defined AC symbols.

Given a finite word problem  $\bigwedge_{i \in I} s_i = t_i \vdash s = t$  where the function symbols are either uninterpreted or AC, AC-completion attempts to transform the conjunction  $\bigwedge_{i \in I} s_i = t_i$  into a finitely terminating, confluent term rewriting system  $R$  whose reductions preserve identity. The rewriting system  $R$  serves as a decision procedure for validating  $s = t$  modulo AC: the equation holds if and only if the normal forms of  $s$  and  $t$  w.r.t  $R$  are equal modulo AC. Furthermore, when its input contains only ground equations, AC-completion terminates and outputs a convergent rewriting system [?].

Unfortunately, AC reasoning is only a part of the automated deduction problem, and what we really need is to decide formulas combining AC symbols and other theories. For instance, in practice, we are interested in deciding finite ground word problems which contain a mixture of uninterpreted, interpreted and AC function symbols, as in the following assertion

$$\begin{aligned} u(a, c_2 - c_1) = a \wedge u(e_1, e_2) - f(b) = u(d, d) \wedge & \vdash a = u(a, 0) \\ d = c_1 + 1 \wedge e_2 = b \wedge u(b, e_1) = f(e_2) \wedge c_2 = 2 * c_1 + 1 & \end{aligned}$$

where  $u$  is an AC symbol,  $+$ ,  $-$ ,  $*$  and the numerals are from the theory of linear arithmetic,  $f$  is an uninterpreted function symbol and the other symbols are uninterpreted constants. A combination of AC reasoning with linear arithmetic and the free theory  $\mathcal{E}$  of equality is necessary to prove this formula. Linear arithmetic is used to show that  $c_2 - c_1 = c_1 + 1$  so that (i)  $u(a, c_1 + 1) = a$  follows by congruence. Independently,  $e_2 = b$  and  $d = c_1 + 1$  imply (ii)  $u(c_1 + 1, c_1 + 1) = 0$  by congruence, linear arithmetic and commutativity of  $u$ . AC reasoning can finally be used to conclude that (i) and (ii) imply that  $u(a, c_1 + 1, c_1 + 1)$  is equal to both  $a$  and  $u(a, 0)$ .

There are two main methods for combining decision procedures for disjoint theories. First, the Nelson-Oppen approach [?] is based on a variable abstraction mechanism and the exchange of equalities between shared variables. Second, the Shostak's algorithm [?] extends a congruence closure procedure with theories equipped with canonizers and solvers, *i.e.* procedures that compute canonical forms of terms and solve equations, respectively. While ground AC-completion can be easily combined with other decision procedures by the Nelson-Oppen method, it cannot be directly integrated in the Shostak's framework since it actually does not provide a solver for the AC theory.

In this paper, we investigate the integration of Shostak theories in ground AC-completion. We first introduce a new notion of rewriting called *canonized* rewriting which adapts normalized rewriting to cope with canonization. Then, we present a modular extension of ground AC-completion for deciding formulas in the combination of the theory of equality with user-defined AC symbols, uninterpreted

<sup>1</sup> Given a term  $t$  of the form  $u(c_1, u(c_2, \dots, u(c_n, c_{n+1}) \dots))$ , the axiomatic approach may have to explicitly handle the  $(2n)!/n!$  terms equivalent to  $t$ .

symbols and an arbitrary signature disjoint Shostak theory  $X$ . The main ideas of our integration are to substitute standard rewriting by canonized rewriting, using a global canonizer for AC and  $X$ , and to replace the equation orientation mechanism found in ground AC-completion with the solver for  $X$ .

AC-completion has been studied for a long time in the rewriting community [?,?]. A generic framework for combining completion with a generic built-in equational theory  $E$  has been proposed in [?]. Normalized completion [?] is designed to use a modified rewriting relation when the theory  $E$  is equivalent to the union of the AC theory and a convergent rewriting system  $\mathcal{S}$ . In this setting, rewriting steps are only performed on  $\mathcal{S}$ -normalized terms.  $AC(X)$  can be seen as an adaptation of ground normalized completion to efficiently handle the theory  $E$  when it is equivalent to the union of the AC theory and a Shostak theory  $X$ . In particular,  $\mathcal{S}$ -normalization is replaced by the application of the canonizer of  $X$ . This modular integration of  $X$  allows us to reuse proof techniques of ground AC-completion [?] to show the correctness of  $AC(X)$ .

Tiwari [?] efficiently combined equality and AC reasoning in the Nelson-Oppen framework. Kapur [?] used ground completion to demystify Shostak's congruence closure algorithm and Bachmair *et al.* [?] compared its strategy with other ones into an abstract congruence closure framework. While the latter approach can also handle AC symbols, none of these works formalized the integration of Shostak theories into (AC) ground completion.

*Outline.* Section 2 recalls standard ground AC completion. Section 3 is devoted to Shostak theories and global canonization. Section 4 presents the  $AC(X)$  algorithm and illustrates its use through an example. The correctness of  $AC(X)$  is detailed in Section 5. In Section 6, we show that a simple preprocessing step allows us to use a partial multiset ordering instead of a full AC-compatible reduction ordering. Experimental results are presented in Section 7. Using a simple example, we show in Section 8 that the instantiation mechanism of ALT-ERGO have to be extended modulo AC in order to fully integrate  $AC(X)$  as the core decision procedure of our SMT solver. Conclusion and future works are presented in Section 9.

## 2 Ground AC-Completion

In this section, we first briefly recall the usual notations and definitions of [?,?] for term rewriting modulo AC. Then, we give the usual set of inference rules for ground AC-completion procedure and we illustrate its use through an example.

Terms are built from a signature  $\Sigma = \Sigma_{AC} \uplus \Sigma_{\mathcal{E}}$  of AC and uninterpreted symbols, and a set of variables  $\mathcal{X}$  yielding the term algebra  $\mathcal{T}_{\Sigma}(\mathcal{X})$ . The range of letters  $a \dots f$  denotes uninterpreted symbols,  $u$  denotes an AC function symbol,  $s, t, l, r$  denote terms, and  $x, y, z$  denote variables. Viewing terms as trees, subterms within a term  $s$  are identified by their positions. Given a position  $p$ ,  $s|_p$  denotes the subterm of  $s$  at position  $p$ , and  $s[r]_p$  the term obtained by replacement of  $s|_p$  by the term  $r$ . We will also use the notation  $s(p)$  to denote the symbol at position  $p$  in the tree, and the root position is denoted by  $\Lambda$ . Given a subset  $\Sigma'$  of  $\Sigma$ , a subterm  $t|_p$  of  $t$  is a  $\Sigma'$ -alien of  $t$  if  $t(p) \notin \Sigma'$  and  $p$  is minimal *w.r.t* the prefix word ordering<sup>2</sup>. We write  $\mathcal{A}_{\Sigma'}(t)$  the multiset of  $\Sigma'$ -aliens of  $t$ .

<sup>2</sup> Notice that according to this definition, a variable may be a  $\Sigma'$ -alien.

A substitution is a partial mapping from variables to terms. Substitutions are extended to a total mapping from terms to terms in the usual way. We write  $t\sigma$  for the application of a substitution  $\sigma$  to a term  $t$ . A well-founded quasi-ordering [?] on terms is a reduction quasi-ordering if  $s \preceq t$  implies  $s\sigma \preceq t\sigma$  and  $l[s]_p \preceq l[t]_p$ , for any substitution  $\sigma$ , term  $l$  and position  $p$ . A quasi-ordering  $\preceq$  defines an equivalence relation  $\simeq$  as  $\preceq \cap \succeq$  and a partial ordering  $\prec$  as  $\preceq \cap \not\succeq$ .

An equation is an unordered pair of terms, written  $s \approx t$ . The variables contained in an equation, if any, are understood as being universally quantified. Given a set of equations  $E$ , the equational theory of  $E$ , written  $=_E$ , is the set of equations that can be obtained by reflexivity, symmetry, transitivity, congruence and instances of equations in  $E^3$ . The word problem for  $E$  consists in determining if, given two ground terms  $s$  and  $t$ , the equation  $s \approx t$  is in  $=_E$ , denoted by  $s =_E t$ . The word problem for  $E$  is ground when  $E$  contains only ground equations. An equational theory  $=_E$  is said to be *inconsistent* when  $s =_E t$ , for *any*  $s$  and  $t$ .

A rewriting rule is an oriented equation, usually denoted by  $l \rightarrow r$ . A term  $s$  rewrites to a term  $t$  at position  $p$  by the rule  $l \rightarrow r$ , denoted by  $s \rightarrow_{l \rightarrow r}^p t$ , iff there exists a substitution  $\sigma$  such that  $s|_p = l\sigma$  and  $t = s[r\sigma]_p$ . A rewriting system  $R$  is a set of rules. We write  $s \rightarrow_R t$  whenever there exists a rule  $l \rightarrow r$  of  $R$  such that  $s$  rewrites to  $t$  by  $l \rightarrow r$  at some position. A normal form of a term  $s$  w.r.t to  $R$  is a term  $t$  such that  $s \rightarrow_R^* t$  and  $t$  cannot be rewritten by  $R$ . The system  $R$  is said to be *convergent* whenever any term  $s$  has a unique normal form, denoted  $s \downarrow_R$ , and does not admit any infinite reduction. Completion [?] aims at converting a set  $E$  of equations into a convergent rewriting system  $R$  such that the sets  $=_E$  and  $\{s \approx t \mid s \downarrow_R = t \downarrow_R\}$  coincide. Given a suitable reduction ordering on terms, it has been proved that completion terminates when  $E$  is ground [?].

*Rewriting modulo AC.* Let  $=_{AC}$  be the equational theory obtained from the set:

$$AC = \bigcup_{u \in \Sigma_{AC}} \{u(x, y) \approx u(y, x), u(x, u(y, z)) \approx u(u(x, y), z)\}$$

In general, given a set  $E$  of equations, it has been shown that no suitable reduction ordering allows completion to produce a convergent rewriting system for  $E \cup AC$ . When  $E$  is ground, an alternative consists in in-lining AC reasoning both in the notion of rewriting step and in the completion procedure.

Rewriting modulo AC is directly related to the notion of matching modulo AC as shown by the following example. Given a rule  $u(a, u(b, c)) \rightarrow t$ , we would like the following reductions to be possible:

$$(1) f(u(c, u(b, a)), d) \rightarrow f(t, d) \quad (2) u(a, u(c, u(d, b))) \rightarrow u(t, d)$$

Associativity and commutativity of  $u$  are needed in (1) for the subterm  $u(c, u(b, a))$  to match the term  $u(a, u(b, c))$ , and in (2) for the term  $u(a, u(c, u(d, b)))$  to be seen as  $u(u(a, u(b, c)), d)$ , so that the rule can be applied. More formally, this leads to the following definition.

**Definition 1 (Ground rewriting modulo AC)** A term  $s$  rewrites to a term  $t$  modulo AC at position  $p$  by the rule  $l \rightarrow r$ , denoted by  $s \rightarrow_{AC \setminus l \rightarrow r}^p t$ , iff (1)  $s|_p =_{AC} l$  and  $t = s[r]_p$  or (2)  $l(A) = u$  and there exists a term  $s'$  such that  $s|_p =_{AC} u(l, s')$  and  $t = s[u(r, s')]_p$

<sup>3</sup> The equational theory of the free theory of equality  $\mathcal{E}$ , defined by the empty set of equations, is simply denoted  $=$ .

In order to produce a convergent rewriting system, ground AC-completion requires a well-founded reduction quasi-ordering  $\preceq$  total on ground terms with an underlying equivalence relation which coincides with  $=_{AC}$ . Such an ordering will be called a total ground AC-reduction ordering.

The inference rules for ground AC-completion are given in Figure 1. The rules describe the evolution of the state of a procedure, represented as a configuration  $\langle E \mid R \rangle$ , where  $E$  is a set of ground equations and  $R$  a ground set of rewriting rules. The initial state is  $\langle E_0 \mid \emptyset \rangle$  where  $E_0$  is a given set of ground equations. **Trivial** removes an equation  $u \approx v$  from  $E$  when  $u$  and  $v$  are equal modulo AC. **Orient** turns an equation into a rewriting rule according to a given total ground AC-reduction ordering  $\preceq$ .  $R$  is used to rewrite either side of an equation (**Simplify**), and to reduce right hand side of rewriting rules (**Compose**). Given a rule  $l \rightarrow r$ , **Collapse** either reduces  $l$  at an inner position, or replaces  $l$  by a term smaller than  $r$ . In both cases, the reduction of  $l$  to  $l'$  may influence the orientation of the rule  $l' \rightarrow r$  which is added to  $E$  as an equation in order to be re-oriented. Finally, **Deduce** adds equational consequences of rewriting rules to  $E$ . For instance, if  $R$  contains two rules of the form  $u(a, b) \rightarrow s$  and  $u(a, c) \rightarrow t$ , then the term  $u(a, u(b, c))$  can either be reduced to  $u(s, c)$  or to the term  $u(t, b)$ . The equation  $u(s, c) \approx u(t, b)$ , called *critical pair*, is thus necessary for ensuring convergence of  $R$ . Critical pairs of a set of rules are computed by the following function ( $a^\mu$  stands for the maximal term w.r.t. size enjoying the assertion):

$$\text{headCP}(R) = \left\{ u(b, r') \approx u(b', r) \mid \begin{array}{l} l \rightarrow r \in R, \quad l' \rightarrow r' \in R \\ \exists a^\mu : l =_{AC} u(a^\mu, b) \wedge l' =_{AC} u(a^\mu, b') \end{array} \right\}$$

<b>Trivial</b>	$\frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\langle E \mid R \rangle} s =_{AC} t$
<b>Orient</b>	$\frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\langle E \mid R \cup \{s \rightarrow t\} \rangle} t \prec s$
<b>Simplify</b>	$\frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\langle E \cup \{s' \approx t\} \mid R \rangle} s \rightarrow_{AC \setminus R} s'$
<b>Compose</b>	$\frac{\langle E \mid R \cup \{l \rightarrow r\} \rangle}{\langle E \mid R \cup \{l \rightarrow r'\} \rangle} r \rightarrow_{AC \setminus R} r'$
<b>Collapse</b>	$\frac{\langle E \mid R \cup \{g \rightarrow d, l \rightarrow r\} \rangle}{\langle E \cup \{l' \approx r\} \mid R \cup \{g \rightarrow d\} \rangle} \begin{cases} l \rightarrow_{AC \setminus g \rightarrow d} l' \\ g \prec l \vee (g \simeq l \wedge d \prec r) \end{cases}$
<b>Deduce</b>	$\frac{\langle E \mid R \rangle}{\langle E \cup \{s \approx t\} \mid R \rangle} s \approx t \in \text{headCP}(R)$

**Fig. 1** Inference rules for ground AC-completion

*Example.* To get a flavor of ground AC-completion, consider a modified version of the assertion given in the introduction, where the arithmetic part has been removed (and uninterpreted constant symbols renamed for the sake of simplicity)

$$u(a_1, a_4) \approx a_1, u(a_3, a_6) \approx u(a_5, a_5), a_5 \approx a_4, a_6 \approx a_2 \vdash a_1 \approx u(a_1, u(a_6, a_3))$$

The precedence  $a_1 \prec_p \dots \prec_p a_6 \prec_p u$  defines an AC-RPO ordering on terms [?] which is suitable for ground AC-completion. The table in Figure 2 shows the application steps of the rules given in Figure 1 from an initial configuration  $\langle \{u(a_1, a_4) \approx a_1, u(a_3, a_6) \approx u(a_5, a_5), a_5 \approx a_4, a_6 \approx a_2\} \mid \emptyset \rangle$  to a final configuration  $\langle \emptyset \mid R_f \rangle$ , where  $R_f$  is the set of rewriting rules  $\{1, 3, 5, 7, 10\}$ . It can be checked that  $a_1 \downarrow_{R_f}$  and  $u(a_1, u(a_6, a_3)) \downarrow_{R_f}$  are identical.

1	$\mathbf{u(a_1, a_4)} \rightarrow \mathbf{a_1}$	<b>Ori</b> $u(a_1, a_4) \approx a_1$
2	$u(a_3, a_6) \rightarrow u(a_5, a_5)$	<b>Ori</b> $u(a_3, a_6) \approx u(a_5, a_5)$
3	$\mathbf{a_5} \rightarrow \mathbf{a_4}$	<b>Ori</b> $a_5 \approx a_4$
4	$u(a_3, a_6) \rightarrow u(a_4, a_4)$	<b>Com</b> 2 and 3
5	$\mathbf{a_6} \rightarrow \mathbf{a_2}$	<b>Ori</b> $a_6 \approx a_2$
6	$u(a_3, a_2) \approx u(a_4, a_4)$	<b>Col</b> 4 and 5
7	$\mathbf{u(a_4, a_4)} \rightarrow \mathbf{u(a_3, a_2)}$	<b>Ori</b> 6
8	$u(a_1, a_4) \approx u(a_1, u(a_3, a_2))$	<b>Ded</b> from 1 and 7
9	$a_1 \approx u(a_1, u(a_3, a_2))$	<b>Sim</b> 8 by 1
10	$\mathbf{u(a_1, u(a_3, a_2))} \rightarrow \mathbf{a_1}$	<b>Ori</b> 9

**Fig. 2** Ground AC-completion example

### 3 Shostak Theories and Global Canonization

In this section, we recall the notions of canonizers and solvers underlying Shostak theories and show how to obtain a global canonizer for the combination of the theories  $\mathcal{E}$  and AC with an arbitrary signature disjoint Shostak theory  $\mathcal{X}$ .

From now on, we assume given a theory  $\mathcal{X}$  with a signature  $\Sigma_{\mathcal{X}}$ . A canonizer for  $\mathcal{X}$  is a function  $\text{can}_{\mathcal{X}}$  that computes a unique normal form for every term such that  $s =_{\mathcal{X}} t$  iff  $\text{can}_{\mathcal{X}}(s) = \text{can}_{\mathcal{X}}(t)$ . A solver for  $\mathcal{X}$  is a function  $\text{solve}_{\mathcal{X}}$  that solves equations between  $\Sigma_{\mathcal{X}}$ -terms. Given an equation  $s \approx t$ ,  $\text{solve}_{\mathcal{X}}(s \approx t)$  either returns a special value  $\perp$  when  $s \approx t \cup \mathcal{X}$  is inconsistent, or an equivalent substitution. A Shostak theory  $\mathcal{X}$  is a theory with a canonizer and a solver which fulfill some standard properties given for instance in [?].

Our combination technique is based on the integration of a Shostak theory  $\mathcal{X}$  in ground AC-completion. From now on, we assume that terms are built from a signature  $\Sigma$  defined as the union of the disjoint signatures  $\Sigma_{AC}$ ,  $\Sigma_{\mathcal{E}}$  and  $\Sigma_{\mathcal{X}}$ . We also assume a total ground AC-reduction ordering  $\preceq$  defined on  $\mathcal{T}_{\Sigma}(\mathcal{X})$  used later on for completion. The combination mechanism requires defining both a global canonizer for the union of  $\mathcal{E}$ , AC and  $\mathcal{X}$ , and a wrapper of  $\text{solve}_{\mathcal{X}}$  to handle heterogeneous equations. These definitions make use of a global one-to-one mapping  $\alpha : \mathcal{T}_{\Sigma} \rightarrow \mathcal{X}$  (and its inverse mapping  $\rho$ ) and are based on a variable abstraction mechanism which computes the *pure*  $\Sigma_{\mathcal{X}}$ -part  $\llbracket t \rrbracket$  of a heterogeneous term  $t$  as follows:

$$\llbracket t \rrbracket = \begin{cases} f(\llbracket \mathbf{s} \rrbracket) & \text{when } t = f(\mathbf{s}) \text{ and } f \in \Sigma_{\mathcal{X}} \\ \alpha(t) & \text{otherwise} \end{cases}$$

The canonizer for AC defined in [?] is based on flattening and sorting techniques which simulate associativity and commutativity, respectively. For instance, the term  $u(u(u'(c, b), b), c)$  is first flattened to  $u(u'(c, b), b, c)$  and then sorted<sup>4</sup> to get the term  $u(b, c, u'(c, b))$ . It has been formally proved that this canonizer solves the word problem for AC [?]. However, this definition implies a modification of the signature  $\Sigma_{AC}$  where arity of AC symbols becomes variadic. Using such canonizer would impact the definition of AC-rewriting given in Section 2. In order to avoid such modification we shall define an equivalent canonizer that builds degenerate trees instead of flattened terms. For instance, we would expect the normal form of  $u(u(u'(c, b), b), c)$  to be  $u(b, u(c, u'(c, b)))$ . Given a signature  $\Sigma$  which contains  $\Sigma_{AC}$  and any total ordering  $\preceq$  on terms, we define  $\text{can}_{AC}$  by:

$$\begin{aligned} \text{can}_{AC}(x) &= x && \text{when } x \in \mathcal{X} \\ \text{can}_{AC}(f(\mathbf{v})) &= f(\text{can}_{AC}(\mathbf{v})) && \text{when } f \notin \Sigma_{AC} \\ \text{can}_{AC}(u(t_1, t_2)) &= u(s_1, u(s_2, \dots, u(s_{n-1}, s_n) \dots)) \\ &\quad \text{where } t'_i = \text{can}_{AC}(t_i) \text{ for } i \in [1, 2] \\ &\quad \text{and } \{\{s_1, \dots, s_n\}\} = \mathcal{A}_{\{u\}}(t'_1) \cup \mathcal{A}_{\{u\}}(t'_2) \\ &\quad \text{and } s_i \preceq s_{i+1} \text{ for } i \in [1, n-1] \text{ when } u \in \Sigma_{AC} \end{aligned}$$

We can easily show that  $\text{can}_{AC}$  enjoys the standard properties required for a canonizer. The proof that  $\text{can}_{AC}$  solves the word problem for AC follows directly from the one given in [?].

Using the technique described in [?], we define our global canonizer  $\text{can}$  which combines  $\text{can}_X$  with  $\text{can}_{AC}$  as follows:

$$\begin{aligned} \text{can}(x) &= x && \text{when } x \in \mathcal{X} \\ \text{can}(f(\mathbf{v})) &= f(\text{can}(\mathbf{v})) && \text{when } f \in \Sigma_{\mathcal{E}} \\ \text{can}(u(s, t)) &= \text{can}_{AC}(u(\text{can}(s), \text{can}(t))) && \text{when } u \in \Sigma_{AC} \\ \text{can}(f_X(\mathbf{v})) &= \text{can}_X(f_X(\llbracket \text{can}(\mathbf{v}) \rrbracket))\rho && \text{when } f_X \in \Sigma_X \end{aligned}$$

Again, the proofs that  $\text{can}$  solves the word problem for the union  $\mathcal{E}$ , AC and  $\mathcal{X}$  and enjoys the standard properties required for a canonizer are similar to those given in [?]. The only difference is that  $\text{can}_{AC}$  directly works on the signature  $\Sigma$ , which avoids the use of a variable abstraction step when canonizing a mixed term of the form  $u(t_1, t_2)$  such that  $u \in \Sigma_{AC}$ .

Using the same mappings  $\alpha$ ,  $\rho$  and the abstraction function, the wrapper  $\text{solve}$  can be easily defined by:

$$\text{solve}(s \approx t) = \begin{cases} \perp & \text{if } \text{solve}_X(\llbracket s \rrbracket \approx \llbracket t \rrbracket) = \perp \\ \{x_i \rho \rightarrow t_i \rho\} & \text{if } \text{solve}_X(\llbracket s \rrbracket \approx \llbracket t \rrbracket) = \{x_i \approx t_i\} \end{cases}$$

In order to ensure termination of  $\text{AC}(X)$ , the global canonizer and the wrapper must be compatible with the ordering  $\preceq$  used by AC-completion, that is:

**Lemma 1**  $\forall t \in \mathcal{T}_{\Sigma}, \text{can}(t) \preceq t$   
 $\forall s, t \in \mathcal{T}_{\Sigma}, \text{if } \text{solve}(s \approx t) = \bigcup \{p_i \rightarrow v_i\} \text{ then } v_i \prec p_i$

We can prove that the above properties hold when the theory  $X$  enjoys the following local compatibility properties:

<sup>4</sup> For instance, using the AC-RPO ordering based on the precedence  $b \prec_p c \prec_p u'$ .

**Axiom 1**  $\forall t \in \mathcal{T}_\Sigma, \text{can}_X(\llbracket t \rrbracket) \preceq \llbracket t \rrbracket$   
 $\forall s, t \in \mathcal{T}_\Sigma, \text{if } \text{solve}_X(\llbracket s \rrbracket \approx \llbracket t \rrbracket) = \bigcup \{x_i \approx t_i\} \text{ then } t_i \rho \prec x_i \rho$

To fulfill this axiom, AC-reduction ordering can be chosen as an AC-RPO ordering [?] based on a precedence relation  $\prec_p$  such that  $\Sigma_X \prec_p \Sigma_{\mathcal{E}} \cup \Sigma_{AC}$ . From now on, we assume that  $X$  is locally compatible with  $\preceq$ .

*Example.* To solve the equation  $u(a, b) + a \approx 0$ , we use the abstraction  $\alpha = \{u(a, b) \mapsto x, a \mapsto y\}$  and call  $\text{solve}_X$  on  $x + y \approx 0$ . Since  $a \prec u(a, b)$ , the only solution which fulfills the axiom above is  $\{x \approx -y\}$ . We apply  $\rho$  and get the set  $\{u(a, b) \rightarrow -a\}$  of rewriting rules.

## 4 Ground AC-Completion Modulo $X$

In this section, we present the  $\text{AC}(X)$  algorithm which extends the ground AC-completion procedure given in Section 2. For that purpose, we first adapt the notion of ground AC-rewriting to cope with canonizers. Then, we show how to refine the inference rules given in Figure 1 to reason modulo the equational theory induced by a set  $E$  of ground equations and the theories  $\mathcal{E}$ , AC and  $X$ .

### 4.1 Canonized Rewriting

From rewriting point of view, a canonizer behaves like a convergent rewriting system: it gives an effective way of computing normal forms. Thus, a natural way for integrating  $\text{can}$  in ground AC-completion is to extend normalized rewriting [?].

**Definition 2** Let  $\text{can}$  be a canonizer. A term  $s$   $\text{can}$ -rewrites to a term  $t$  at position  $p$  by the rule  $l \rightarrow r$ , denoted by  $s \rightsquigarrow_{l \rightarrow r}^p t$ , iff

$$s \rightarrow_{AC \setminus l \rightarrow r}^p t' \quad \text{and} \quad \text{can}(t') = t$$

*Example.* Using the usual canonizer  $\text{can}_{\mathcal{A}}$  for linear arithmetic and the rule  $\gamma : u(a, b) \rightarrow a$ , the term  $f(a + 2 * u(b, a))$   $\text{can}_{\mathcal{A}}$ -rewrites to  $f(3 * a)$  by  $\rightsquigarrow_\gamma$  as follows:  $f(a + 2 * u(b, a)) \rightarrow_{AC \setminus \gamma} f(a + 2 * a)$  and  $\text{can}_{\mathcal{A}}(f(a + 2 * a)) = f(3 * a)$ .

**Lemma 2**  $\forall s, t. s \rightsquigarrow_{l \rightarrow r} t \implies s =_{AC, X, l \approx r} t$

### 4.2 The $\text{AC}(X)$ Algorithm

The first step of our combination technique consists in replacing the rewriting relation found in completion by canonized rewriting. This leads to the rules of  $\text{AC}(X)$  given in Figure 3. The state of the procedure is a pair  $\langle E \mid R \rangle$  of equations and rewriting rules. The initial configuration is  $\langle E_0 \mid \emptyset \rangle$  where  $E_0$  is supposed to be a set of equations between canonized terms. Since  $\text{AC}(X)$ 's rules only involve canonized rewriting, the algorithm maintains the invariant that terms occurring in  $E$  and  $R$  are in canonical forms. **Trivial** thus removes an equation  $u \approx v$  from  $E$  when  $u$  and  $v$  are syntactically equal. A new rule **Bottom** is used to detect inconsistent equations. Similarly to normalized completion, integrating the global



canonizer `can` in rewriting is not enough to fully extend ground AC-completion with the theory  $X$ : in both cases the orientation mechanism has to be adapted. Therefore, the second step consists in integrating the wrapper `solve` in the **Orient** rule. The other rules are much similar to those of ground AC-completion except that they use the relation  $\rightsquigarrow_R$  instead of  $\rightarrow_{AC \setminus R}$ .

$\mathbf{Trivial} \frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\langle E \mid R \rangle} s = t \quad \mathbf{Bottom} \frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\perp} \text{solve}(s, t) = \perp$
$\mathbf{Orient} \frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\langle E \mid R \cup \text{solve}(s, t) \rangle} \text{solve}(s, t) \neq \perp$
$\mathbf{Simplify} \frac{\langle E \cup \{s \approx t\} \mid R \rangle}{\langle E \cup \{s' \approx t\} \mid R \rangle} s \rightsquigarrow_R s' \quad \mathbf{Compose} \frac{\langle E \mid R \cup \{l \rightarrow r\} \rangle}{\langle E \mid R \cup \{l \rightarrow r'\} \rangle} r \rightsquigarrow_R r'$
$\mathbf{Collapse} \frac{\langle E \mid R \cup \{g \rightarrow d, l \rightarrow r\} \rangle}{\langle E \cup \{l' \approx r\} \mid R \cup \{g \rightarrow d\} \rangle} \begin{cases} l \rightsquigarrow_{g \rightarrow d} l' \\ g \prec l \vee (g \simeq l \wedge d \prec r) \end{cases}$
$\mathbf{Deduce} \frac{\langle E \mid R \rangle}{\langle E \cup \{s \approx t\} \mid R \rangle} s \approx t \in \text{headCP}(R)$

**Fig. 3** Inference rules for ground AC-completion modulo  $X$

*Example.* We illustrate  $AC(X)$  on the example given in the introduction:

$$\begin{aligned} u(a, c_2 - c_1) \approx a \wedge u(e_1, e_2) - f(b) \approx u(d, d) \wedge & \vdash a \approx u(a, 0) \\ d \approx c_1 + 1 \wedge e_2 \approx b \wedge u(b, e_1) \approx f(e_2) \wedge c_2 \approx 2 * c_1 + 1 & \end{aligned}$$

The table given in Figure 4 shows the application of the rules of  $AC(X)$  on the example when  $X$  is instantiated by linear arithmetic. We use an AC-RPO ordering based on the precedence  $1 \prec_p 2 \prec_p a \prec_p b \prec_p c_1 \prec_p c_2 \prec_p d \prec_p e_1 \prec_p e_2 \prec_p f \prec_p u$ . The procedure terminates and produces a convergent rewriting system  $R_f = \{3, 5, 9, 10, 11, 13, 16\}$ . Using  $R_f$ , we can check that  $a$  and  $u(a, 0)$  can-rewrite to the same normal form.

## 5 Correctness

In this section, we give detailed proofs for the correctness of  $AC(X)$ . This property is stated by the theorem below and its proof is based on three intermediate theorems, stating respectively soundness, completeness and termination.

As usual, in order to enforce correctness, we cannot use any (unfair) strategy. We say that a strategy is *strongly fair* when no possible application of an inference rule is infinitely delayed and **Orient** is only applied over fully reduced terms.

**Theorem 1** *Given a set  $E$  of ground equations, the application of the rules of  $AC(X)$  under a strongly fair strategy terminates and either produces  $\perp$  when  $E \cup AC \cup X$  is inconsistent, or yields a final configuration  $\langle \emptyset \mid R \rangle$  such that:  $\forall s, t \in \mathcal{T}_\Sigma. s =_{E, AC, X} t \iff \text{can}(s) \downarrow_R = \text{can}(t) \downarrow_R$*

1	$u(a, c_2 - c_1) \rightarrow a$	<b>Ori</b> $u(a, c_2 - c_1) \approx a$
2	$u(e_1, e_2) \rightarrow u(d, d) + f(b)$	<b>Ori</b> $u(e_1, e_2) - f(b) \approx u(d, d)$
3	$\mathbf{d} \rightarrow \mathbf{c}_1 + \mathbf{1}$	<b>Ori</b> $d \approx c_1 + 1$
4	$u(e_1, e_2) \rightarrow u(c_1 + 1, c_1 + 1) + f(b)$	<b>Com</b> 2 and 3
5	$\mathbf{e}_2 \rightarrow \mathbf{b}$	<b>Ori</b> $e_2 \approx b$
6	$u(b, e_1) \approx u(c_1 + 1, c_1 + 1) + f(b)$	<b>Col</b> 4 and 5
7	$u(b, e_1) \rightarrow u(c_1 + 1, c_1 + 1) + f(b)$	<b>Ori</b> $u(b, e_1) \approx u(c_1 + 1, c_1 + 1) + f(b)$
8	$u(c_1 + 1, c_1 + 1) + f(b) \approx f(b)$	<b>Sim</b> $u(b, e_1) \approx f(e_2)$ by 5 and 7
9	$\mathbf{u}(\mathbf{c}_1 + \mathbf{1}, \mathbf{c}_1 + \mathbf{1}) \rightarrow \mathbf{0}$	<b>Ori</b> $u(c_1 + 1, c_1 + 1) + f(b) \approx f(b)$
10	$\mathbf{u}(\mathbf{b}, \mathbf{e}_1) \rightarrow \mathbf{f}(\mathbf{b})$	<b>Com</b> 7 and 9
11	$\mathbf{c}_2 \rightarrow 2 * \mathbf{c}_1 + \mathbf{1}$	<b>Ori</b> $c_2 \approx 2 * c_1 + 1$
12	$u(a, c_1 + 1) \approx a$	<b>Col</b> 1 and 11
13	$\mathbf{u}(\mathbf{a}, \mathbf{c}_1 + \mathbf{1}) \rightarrow \mathbf{a}$	<b>Ori</b> $u(a, c_1 + 1) \approx a$
14	$u(0, a) \approx u(a, c_1 + 1)$	<b>Ded</b> from 9 and 13
15	$u(0, a) \approx a$	<b>Sim</b> 14 by 13
16	$\mathbf{u}(\mathbf{0}, \mathbf{a}) \rightarrow \mathbf{a}$	<b>Ori</b> 15

**Fig. 4** AC(X) on the running example.

In the following, we shall consider a fixed run of the completion procedure

$$\langle E_0 \mid \emptyset \rangle \rightarrow \langle E_1 \mid R_1 \rangle \rightarrow \dots \rightarrow \langle E_n \mid R_n \rangle \rightarrow \langle E_{n+1} \mid R_{n+1} \rangle \rightarrow \dots$$

starting from the initial configuration  $\langle E_0 \mid \emptyset \rangle$ . We denote  $R_\infty$  (resp.  $E_\infty$ ) the set of all encountered rules  $\bigcup_n R_n$  (resp. equations  $\bigcup_n E_n$ ) and  $\mathcal{R}_\omega$  (resp.  $E_\omega$ ) the set of persistent rules  $\bigcup_n \bigcap_{i \geq n} R_i$  (resp. equations  $\bigcup_n \bigcap_{i \geq n} E_i$ ).

The strongly fair strategy requirement implies in particular that  $\text{headCP}(R_\omega) \subseteq E_\infty$ ,  $E_\omega = \emptyset$  and  $R_\omega$  is inter-reduced, that is none of its rules can be collapsed or composed by another one. Due to the assumptions made over  $\text{can}_X$  and  $\prec$ , the following valid properties will be continuously used in the proofs:

$$\begin{aligned} \forall t, \text{can}(t) &\preceq t \\ \forall s, t, s \simeq t &\iff s =_{AC} t \\ \forall s, t, s \rightsquigarrow_{R_\infty} t &\implies t \prec s \end{aligned}$$

### 5.1 Soundness

The soundness property of AC(X) is ensured by the following invariant:

**Theorem 2** For any configuration  $\langle E_n \mid R_n \rangle$  reachable from  $\langle E_0 \mid \emptyset \rangle$ ,

$$\forall s, t, (s, t) \in E_n \cup R_n \implies s =_{AC, X, E_0} t$$

*Proof* The invariant obviously holds for the initial configuration and is preserved by all the inference rules. The rules **Simplify**, **Compose**, **Collapse** and **Deduce** preserve the invariant since for any rule  $l \rightarrow r$ , if  $l =_{AC, X, E_0} r$ , for any term  $s$  rewritten by  $\rightsquigarrow_{l \rightarrow r}$  into  $t$ , then  $s =_{AC, X, E_0} t$ . If **Orient** is used to turn an equation

$s \approx t$  into a set of rules  $\{p_i \rightarrow v_i\}$ , by definition of `solve`,  $p_i = x_i\rho$  and  $v_i = t_i\rho$ , where  $\text{solve}_X(\llbracket s \rrbracket \approx \llbracket t \rrbracket) = \{x_i \approx t_i\}$ . By soundness of `solve`  $x_i =_{X, \llbracket s \rrbracket \approx \llbracket t \rrbracket} t_i$ . An equational proof of  $x_i =_{X, \llbracket s \rrbracket \approx \llbracket t \rrbracket} t_i$  can be instantiated by  $\rho$ , yielding an equational proof  $p_i =_{X, s \approx t} v_i$ . Since by induction  $s =_{AC, X, E_0} t$  holds, we get  $p_i =_{AC, X, E_0} v_i$ .

## 5.2 Completeness

Completeness is established in several steps using a variant of the technique introduced by Bachmair *et al.* in [?] for proving completeness of completion. This technique transforms a proof between two terms which is not under a suitable form into a smaller one, and the smallest proofs are the desired ones.

The proofs we are considering are made of elementary steps, either equational steps, with AC, X and  $E_\infty$ , or rewriting steps, with  $R_\infty$  and the additional (possibly infinite) rules  $R_{\text{can}} = \{t \rightarrow \text{can}(t) \mid \text{can}(t) \neq t\}$ . Rewriting steps with  $R_\infty$  can be either  $\rightsquigarrow_{R_\infty}$  or  $\rightarrow_{R_\infty}$ <sup>5</sup>.

The measure of a proof is the multiset of the elementary measures of its elementary steps. The measure of an elementary step is a 5-tuple of type

$$\text{multiset}(\mathcal{T}_\Sigma(\mathcal{X})) \times \mathbb{N} \times \mathbb{N} \times \mathcal{T}_\Sigma(\mathcal{X}) \times \mathcal{T}_\Sigma(\mathcal{X})$$

It takes into account the number of terms which are in a canonical form in an elementary proof: the canonical weight of a term  $t$ ,  $w_{\text{can}}(t)$  is equal to 0 if  $\text{can}(t) =_{AC} t$  and to 1 otherwise. Notice that if  $w_{\text{can}}(t) = 1$ , then  $\text{can}(t) \prec t$ , and if  $w_{\text{can}}(t) = 0$ , then  $\text{can}(t) \simeq t$ . The measure of an elementary step between  $t_1$  and  $t_2$  is performed thanks to

- an equation is equal to  $(\{\{t_1, t_2\}\}, \rightarrow, \rightarrow, \rightarrow, -)$
- a rule  $l \rightarrow r \in R_\infty$  is equal to

$$(\{\{t_1\}\}, 1, w_{\text{can}}(t_1) + w_{\text{can}}(t_2), l, r)$$

if  $t_1 \rightsquigarrow_{l \rightarrow r} t_2$  (or  $t_1 \rightarrow_{l \rightarrow r} t_2$ ), and to

$$(\{\{t_2\}\}, 1, w_{\text{can}}(t_1) + w_{\text{can}}(t_2), l, r)$$

if  $t_1 \leftarrow_{r \leftarrow l} t_2$  (or  $t_1 \leftarrow_{r \leftarrow l} t_2$ ). In the case of a  $\rightsquigarrow$  step, the measure is actually  $(\{\{t_i\}\}, 1, w_{\text{can}}(t_i), l, r)$  since the reduct is always in a canonical form.

- a rule of  $R_{\text{can}}$  is equal to

$$(\{\{t_1\}\}, 0, w_{\text{can}}(t_1) + w_{\text{can}}(t_2), t_1, t_2)$$

if  $t_1 \rightarrow_{R_{\text{can}}} t_2$ , and to

$$(\{\{t_2\}\}, 0, w_{\text{can}}(t_1) + w_{\text{can}}(t_2), t_2, t_1)$$

if  $t_1 \leftarrow_{R_{\text{can}}} t_2$ .

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<sup>5</sup> Here,  $s \rightarrow_{R_\infty} t$  actually means  $s \rightarrow_{AC \setminus R_\infty} t'$  and  $t = \text{can}_{AC}(t')$ .

Elementary steps are compared lexicographically using the multiset extension of  $\preceq$  for the first component, the usual ordering over natural numbers for the components 2 and 3, and  $\preceq$  for last ones. Since  $\preceq$  is an AC-reduction ordering, the ordering defined over proofs is well-founded.

The general methodology is to show that a proof which contains some unwanted elementary steps can be replaced by a proof with a strictly smaller measure. Since the ordering over measures is well-founded, there exists a minimal proof, and such a minimal proof is of the desired form.

**Lemma 3** *A proof containing an elementary step  $\longleftrightarrow_{s \approx t}$ , where  $s \approx t \in AC \cup X$  is not minimal.*

*Proof* An elementary equational step using an equation  $s \approx t$  of  $AC \cup X$  under the context  $C[-]_p$  can be reduced: the subproof

$$C[s]_p \xleftrightarrow[s \approx t]{} C[t]_p$$

is replaced by

$$C[s]_p \xrightarrow[R_{\text{can}}]{\{0,1\}} \text{can}(C[s]_p) = \text{can}(C[t]_p) \xleftarrow[R_{\text{can}}]{\{0,1\}} C[t]_p$$

The measure strictly decreases, since for the first subproof it is equal to  $\{\{\{C[s]_p, C[t]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}\}$ , and for the second one, it is equal to  $\{\{\{C[s]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}^{\{0,1\}}, \{\{C[t]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}^{\{0,1\}}\}$ . The rewrite steps  $\rightarrow_{R_{\text{can}}}^{\{0,1\}}$  only occur on a term which is not AC-equal to a canonical form (which is denoted by the  $\{0,1\}$  exponent). The corresponding elementary measure occurs in the global measure of the second subproof accordingly.

**Lemma 4** *A proof containing an elementary step  $\longleftrightarrow_{s \approx t}$ , where  $s \approx t \in E_\infty$  is not minimal.*

*Proof* An elementary equational step using an equation  $s \approx t$  of  $E_\infty$  under the context  $C[-]_p$  can be reduced. Since  $E_\omega$  is empty, there is a completion state where  $s \approx t$  disappears, either by **Simplify** or **Orient**.

- If **Simplify** is used to reduce  $s$  into  $s'$  by the rule  $l \rightarrow r$  of  $R_\infty$ , the subproof

$$C[s]_p \xleftrightarrow[s \approx t]{} C[t]_p$$

is replaced by

$$C[s]_p \xrightarrow[l \rightarrow r]{} C[s']_p \xleftrightarrow[s' \approx t]{} C[t]_p$$

The measure strictly decreases, since for the first subproof it is equal to  $\{\{\{C[s]_p, C[t]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}\}$ , and for the second one, it is equal to  $\{\{\{C[s]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}, \{\{C[s']_p, C[t]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}\}$ , and  $s \succ s'$ .

- If the rule **Orient** turns  $s \approx t$  into a set of rules  $\pi = \{p_i \rightarrow v_i\}$ , by definition of **solve** we have  $\text{solve}_X(\llbracket s \rrbracket \approx \llbracket t \rrbracket) = \{x_i \approx t_i\}$  (denoted as  $\sigma$ ) with  $p_i = x_i \rho$  and  $v_i = t_i \rho$ . Since  $\text{solve}_X$  is complete,  $\llbracket s \rrbracket \sigma =_X \llbracket t \rrbracket \sigma$ . Consider a variable  $x$  of  $\llbracket s \rrbracket$  or  $\llbracket t \rrbracket$ ,
  - if  $x \in \{x_i\}$  then  $x \rho \pi = p_i \pi = v_i$  and  $x \sigma \rho = t_i \rho = v_i$ .
  - if  $x \notin \{x_i\}$  then  $x \rho \pi = x \rho$  (since  $x \rho \notin \{p_i\}$ ) and  $x \sigma \rho = x \rho$  (since  $x \sigma = x$ ).

In all cases,  $x\rho\pi = x\sigma\rho$ . The equational step using  $s \approx t$  can be recovered as a compound step using  $\pi$  and  $R_{\text{can}}$  as follows:

$$\begin{aligned} C[s]_p &= C[[s]\rho]_p \xrightarrow{\pi^+} \\ C[[s]\rho\pi]_p &= C[[s]\sigma\rho]_p \xrightarrow{R_{\text{can}}^{0,1}} \xleftarrow{R_{\text{can}}^{0,1}} C[[t]\sigma\rho]_p = C[[t]\rho\pi]_p \\ &\xleftarrow{\pi^+} C[[t]\rho]_p = C[t]_p \end{aligned}$$

The set of rules  $\pi$  belongs to  $R_\infty$ , and the measure of the new subproof is a multiset containing only elements of the form  $(\{\{C[s_i]_p\}, -, -, -, -\})$ , where  $s_i$  is a reduct of a subterm  $s$  or  $t$  by an arbitrary number of steps of  $R_\infty$  and  $R_{\text{can}}$ . In any case,  $\{\{C[s_i]_p\}\} \prec \{\{C[s]_p, C[t]_p\}\}$ . The new subproof is strictly smaller than the measure of the original subproof.

**Lemma 5** *A proof containing an elementary rewriting step truly of the form  $\rightarrow_{R_\infty}$  or  $\leftarrow_{R_\infty}$  is not minimal.*

*Proof* Here, each elementary step  $s \rightarrow_{R_\infty} t$  is already a  $\rightsquigarrow_{R_\infty}$  step if  $t = \text{can}_{AC}(t)$  is in a canonical form w.r.t  $\text{can}$ , or it can be replaced by

$$s \rightsquigarrow_{R_\infty} \text{can}(t) \xleftarrow{R_{\text{can}}} t$$

The measure of the first subproof is equal to  $\{\{\{s\}, 1, w_{\text{can}}(s) + w_{\text{can}}(t), -, -, -\}\}$ , and the measure of the second one is equal to  $\{\{\{s\}, 1, w_{\text{can}}(s), -, -, -\}, (\{t\}, 0, -, -, -)\}$ , and  $t \prec s$ . Since  $w_{\text{can}}(t) = 1$ , the measure strictly decreases.

The case  $s \leftarrow_{R_\infty} t$  is symmetrical.

**Lemma 6** *A proof containing an elementary rewriting step of the form  $\rightsquigarrow_{l \rightarrow r}$  or  $\leftarrow_{r \leftarrow l}$ , where  $l \rightarrow r \in R_\infty \setminus R_\omega$  is not minimal.*

*Proof* An elementary  $\rightsquigarrow$  step using a rule  $l \rightarrow r$  of  $R_\infty \setminus R_\omega$  can be reduced. The rule  $l \rightarrow r$  disappears either by **Compose** or by **Collapse**.

– If **Compose** reduces  $r$  to  $r' = \text{can}(r[d])$  by the rule  $g \rightarrow d$  of  $R_\infty$ , the subproof

$$C[l]_p \rightsquigarrow_{l \rightarrow r} \text{can}(C[r]_p)$$

can be replaced by

$$C[l]_p \rightsquigarrow_{l \rightarrow r'} \text{can}(C[r']_p) = \text{can}(C[r[d]]_p) \xleftarrow{d \leftarrow g} C[r]_p$$

The identity  $\text{can}(C[r']_p) = \text{can}(C[r[d]]_p)$  holds  $C[r']_p$  and  $C[r[d]]_p$  are equal modulo  $R_{\text{can}}$ , that is  $AC \cup X$ , and such terms have the same canonical forms. The measure strictly decreases, since for the first subproof it is equal to  $\{\{\{C[l]_p\}, 1, w_{\text{can}}(C[l]_p), l, r\}\}$ , and for the second one, it is equal to  $\{\{\{C[l]_p\}, 1, w_{\text{can}}(C[l]_p), l, r'\}\}, (\{C[r]_p\}, 0, -, -, -)\}$ , and  $r' \prec r \prec l$ .

– If **Collapse** reduces  $l$  to  $l' = \mathbf{can}(l[d])$  by the rule  $g \rightarrow d$  in  $R_\infty$ , the subproof

$$C[l]_p \rightsquigarrow_{l \rightarrow r} \mathbf{can}(C[r]_p)$$

is replaced by

$$C[l]_p \rightsquigarrow_{g \rightarrow d} \mathbf{can}(C[l[d]]_p) = \mathbf{can}(C[l']_p) \xleftarrow{R_{\mathbf{can}}} C[l']_p \xleftrightarrow{l' \approx r} C[r]_p \xrightarrow{R_{\mathbf{can}}} \mathbf{can}(C[r]_p)$$

The measure strictly decreases, since for the first subproof it is equal to  $\{\{C[l]_p\}, 1, w_{\mathbf{can}}(C[l]_p), l, r\}$ , and for the second one, it is equal to

$$\{\{C[l]_p\}, 1, w_{\mathbf{can}}(C[l]_p), g, d\}, \\ \{\{C[l']_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}, \{\{C[l']_p C[r]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}, \{\{C[r]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}$$

The last three elements of the second multiset are strictly smaller than the element of the first multiset, since  $l' \prec l$  and  $r \prec l$ . The first element of the second multiset is strictly smaller than the element of the first multiset, since either  $g \prec l$ , and the fourth component decreases, or  $g \simeq l$  and  $d \prec g$ . In this case,  $l' = d \prec r$ . The first four components are identical, and the last one decreases.

The case  $\leftarrow$  is symmetrical.

**Lemma 7** *A proof containing a peak  $s \leftarrow_{R_{\mathbf{can}}} t \rightarrow_{R_{\mathbf{can}}} s'$  is not minimal.*

*Proof* All the terms  $s, t$  and  $s'$  involved in the peak are equal modulo AC and X, hence  $\mathbf{can}(s) = \mathbf{can}(s')$ . The subproof

$$s \leftarrow_{R_{\mathbf{can}}} t \rightarrow_{R_{\mathbf{can}}} s'$$

is replaced by

$$s \xrightarrow{R_{\mathbf{can}}^{\{0,1\}}} \mathbf{can}(s) = \mathbf{can}(s') \xleftarrow{R_{\mathbf{can}}^{\{0,1\}}} s'$$

The measure strictly decreases, since for the first subproof it is equal to

$$\{\{t\}, 0, w_{\mathbf{can}}(t) + w_{\mathbf{can}}(s), \rightarrow, \rightarrow\}, \{\{t\}, 0, w_{\mathbf{can}}(t) + w_{\mathbf{can}}(s'), \rightarrow, \rightarrow\}$$

and for the second one, it is equal to

$$\{\{s\}, 0, w_{\mathbf{can}}(s), \rightarrow, \rightarrow\}^{\{0,1\}}, \{\{s'\}, 0, w_{\mathbf{can}}(s), \rightarrow, \rightarrow\}^{\{0,1\}}.$$

$s$  and  $s'$  are smaller than or equivalent to  $t$  ( $s, s' \preceq t$ ), and the second component strictly decreases, since  $\mathbf{can}(s)$  and  $\mathbf{can}(s')$  are in a canonical form and  $t$  is not.

**Lemma 8** *A proof containing a peak  $s \leftarrow_{R_\omega} t \rightsquigarrow_{R_\omega} s'$  is not minimal.*

*Proof* We make a case analysis over the positions of the reductions.

– In the parallel case, the subproof

$$s \begin{array}{c} \xleftarrow{p} t \xrightarrow{q} s' \\ r \leftarrow l \quad g \rightarrow d \end{array}$$

can be seen as

$$s = \mathbf{can}(t[r]_p[g]_q) \xleftarrow{R_{\mathbf{can}}} t[r]_p[g]_q \xleftarrow{r \leftarrow l} t[l]_p[g]_q \xrightarrow{g \rightarrow d} t[l]_p[d]_q \xrightarrow{R_{\mathbf{can}}} \mathbf{can}(t[l]_p[d]_q) = s'$$

The above subproof can be replaced by

$$s = \mathbf{can}(t[r]_p[g]_q) \xleftarrow{R_{\mathbf{can}}^{\{0,1\}}} t[r]_p[g]_q \xrightarrow{g \rightarrow d} \mathbf{can}(t[r]_p[d]_q) \xleftarrow{r \leftarrow l} t[l]_p[d]_q \xrightarrow{R_{\mathbf{can}}^{\{0,1\}}} \mathbf{can}(t[l]_p[d]_q) = s'$$

The measure strictly decreases, since for the first subproof it is equal to  $\{\{t\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}$ ,  $(\{t\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow)$ , and for the second one, it is equal to

$$\{\{t[r]_p[g]_q\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}^{\{0,1\}}, (\{t[r]_p[g]_q\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow), \\ (\{t[l]_p[d]_q\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow), (\{t[l]_p[d]_q\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow)^{\{0,1\}}$$

and both terms  $t[r]_p[g]_q$  and  $t[l]_p[d]_q$  are strictly smaller than  $t = t[l]_p[g]_q$ .

- If  $q$  is a strict prefix of  $p$ , this means that  $l \rightarrow r$  can be used to collapse the rule  $g \rightarrow d$ , which is impossible since the strategy is strongly fair, and the application of **Collapse** cannot be infinitely delayed.
- The case where  $p$  is a strict prefix of  $q$  is similar.
- If  $p$  and  $q$  are equal, this means that in both reductions, the extended rewriting has been used (second case of definition 1). Otherwise, again, one rule could collapse the other. This means that  $l$  and  $g$  have the same AC top function symbol  $u$ . When  $l$  and  $g$  do not share a common subterm, the reasoning is similar to the parallel case. Otherwise, if they share a common subterm, since the strategy is fair, the head critical pair between  $l \rightarrow r$  and  $g \rightarrow d$  has been computed. Let  $a^\mu$  the maximal common part between  $l$  and  $g$ ,  $l =_{AC} u(a^\mu, b)$ , and  $g =_{AC} u(a^\mu, b')$ . The critical pair is  $u(b', r) \approx u(b, d)$ . The subterm  $t|_p$  where both reductions occur is of the form  $u(a^\mu, u(b, u(b', c)))$  (or  $u(a^\mu, u(b, b'))$ ) if it corresponds exactly to the critical pair).

The subproof can be replaced by

$$s = \xleftarrow{R_{\mathbf{can}}} t[u(u(b', r), c)]_p \xrightarrow{u(b', r) \approx u(b, d)} t[u(u(b, d), c)]_p \xrightarrow{R_{\mathbf{can}}} s'$$

The measure strictly decreases, since for the first subproof it is equal to  $\{\{t\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}$ ,  $(\{t\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow)$ , and for the second one, it is equal to

$$\{\{t[u(u(b', r), c)]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}, \{\{t[u(u(b', r), c)]_p, t[u(u(b, d), c)]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}, \\ \{\{t[u(u(b, d), c)]_p\}, \rightarrow, \rightarrow, \rightarrow, \rightarrow\}$$

and both  $t[u(u(b', r), c)]_p$  and  $t[u(u(b, d), c)]_p$  are strictly smaller than  $t$ .

**Lemma 9** *A proof containing a peak  $s \xleftarrow{R_\omega} t \xrightarrow{R_{\mathbf{can}}} s'$  is not minimal.*

The proof of this lemma is partly made by structural induction over  $t$ , and we need an auxiliary result in order to study how behave a proof plugged under a context.

**Definition 3** Given a context  $C[\bullet]_p$ , and an elementary proof  $\mathcal{P}$ ,  $\mathcal{P}$  plugged under  $C[\bullet]_p$ , denoted as  $C[\mathcal{P}]_p$  is defined as follows:

- if  $\mathcal{P}$  is an equational step  $s \leftrightarrow_{l \approx r} t$ ,  $C[\mathcal{P}]_p$  is  $C[s]_p \leftrightarrow_{l \approx r} C[t]_p$ .
- if  $\mathcal{P}$  is a rewriting step  $s \rightarrow_{l \rightarrow r} t$ ,  $C[\mathcal{P}]_p$  is  $C[s]_p \rightarrow_{l \rightarrow r} C[t]_p$ .
- if  $\mathcal{P}$  is a rewriting step  $s \rightsquigarrow_{l \rightarrow r} t$ ,  $C[\mathcal{P}]_p$  is either

$$C[s]_p \rightsquigarrow_{l \rightarrow r} \mathbf{can}(C[t]_p) \leftarrow_{R_{\mathbf{can}}}^A C[t]_p$$

if  $C[t]_p$  is not in a canonical form, or  $C[s]_p \rightsquigarrow_{l \rightarrow r} \mathbf{can}(C[t]_p)$  otherwise.

This definition is extended to a proof made of several steps, by plugging elementary each step under the context. Notice that if a proof  $\mathcal{P}$  relates two terms  $s$  and  $t$ , then  $C[\mathcal{P}]_p$  relates  $C[s]_p$  and  $C[t]_p$ .

**Lemma 10** *Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two proofs which do not contain  $\rightarrow_{R_\infty}$  nor  $\leftarrow_{R_\infty}$ . If  $\mathcal{P}_1$  is strictly smaller than (resp. equivalent to)  $\mathcal{P}_2$ , then  $C[\mathcal{P}_1]_p$  is strictly smaller than (resp. equivalent to)  $C[\mathcal{P}_2]_p$ . Moreover if  $\mathcal{P}_2$  is a step  $s \rightsquigarrow_{l \rightarrow r} t$ ,  $C[\mathcal{P}_1]_p$  is strictly smaller than  $C[s]_p \rightsquigarrow_{l \rightarrow r} C[t]_p$*

*Proof* It is enough to show the wanted result for elementary steps. Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two elementary steps such that  $\mathcal{P}_1$  is strictly smaller than  $\mathcal{P}_2$ .

- If  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are  $\rightarrow_{R_{\mathbf{can}}}$  steps, they are of the form

$$s_i \xrightarrow{R_{\mathbf{can}}} t_i$$

and the corresponding measures are  $(\{\{s_i\}\}, 0, w_{\mathbf{can}}(s_i) + w_{\mathbf{can}}(t_i), s_i, t_i)$ .

- if  $s_1 \prec s_2$ , then  $C[s_1]_p \prec C[s_2]_p$ .
- if  $s_1 \simeq s_2$ , and  $w_{\mathbf{can}}(s_1) + w_{\mathbf{can}}(t_1) < w_{\mathbf{can}}(s_2) + w_{\mathbf{can}}(t_2)$ . Since  $s_1 \simeq s_2$ , by the AC-totality of  $\preceq$ , we know that  $s_1 =_{AC} s_2$ , hence  $w_{\mathbf{can}}(s_1) = w_{\mathbf{can}}(s_2)$ . This means that  $w_{\mathbf{can}}(t_1) = 0$  and  $w_{\mathbf{can}}(t_2) = 1$ . Hence  $t_1 =_{AC} \mathbf{can}(t_1)$ ,  $t_1 \simeq \mathbf{can}(t_1)$  and  $t_2 \neq_{AC} \mathbf{can}(t_2)$  and  $\mathbf{can}(t_2) \prec t_2$ . Since  $s_1 =_{AC} s_2$ ,  $\mathbf{can}(t_1) = \mathbf{can}(t_2)$  holds, hence  $t_1 \prec t_2$ .  
If we look at the plugged proofs, we have  $C[s_1]_p \simeq C[s_2]_p$ ,  $w_{\mathbf{can}}(C[s_1]_p) = w_{\mathbf{can}}(C[s_2]_p)$ ,  $w_{\mathbf{can}}(C[t_1]_p) \leq w_{\mathbf{can}}(C[t_2]_p) = 1$  and  $C[t_1]_p \prec C[t_2]_p$ . The measure is even on the first component, and either strictly decreases on the second component, or weakly decreases over the four first components, and strictly decreases over the last one. In all cases,  $C[\mathcal{P}_1]_p$  is strictly smaller than  $C[\mathcal{P}_2]_p$ .
- if  $s_1 \simeq s_2$  and  $w_{\mathbf{can}}(s_1) + w_{\mathbf{can}}(t_1) = w_{\mathbf{can}}(s_2) + w_{\mathbf{can}}(t_2)$ , this means that  $t_1 \prec t_2$ . The case  $w_{\mathbf{can}}(t_1) = w_{\mathbf{can}}(t_2) = 0$  is impossible, since this would imply  $t_1 \simeq \mathbf{can}(t_1) = \mathbf{can}(t_2) \simeq t_2$ . Hence  $w_{\mathbf{can}}(t_1) = w_{\mathbf{can}}(t_2) = 1$ .  
If we look at the plugged proofs, we have  $C[s_1]_p \simeq C[s_2]_p$ ,  $w_{\mathbf{can}}(C[s_1]_p) = w_{\mathbf{can}}(C[s_2]_p)$ ,  $w_{\mathbf{can}}(C[t_1]_p) = w_{\mathbf{can}}(C[t_2]_p) = 1$  and  $C[t_1]_p \prec C[t_2]_p$ . The measure is even on the first four components, and strictly decreases over the last one.  $C[\mathcal{P}_1]_p$  is strictly smaller than  $C[\mathcal{P}_2]_p$ .
- if  $\mathcal{P}_1$  is a  $\rightsquigarrow$ -step, and  $\mathcal{P}_2$  is a  $\rightarrow_{R_{\mathbf{can}}}$  step, necessarily, the first component strictly decreases. The measure of  $C[\mathcal{P}_1]_p$  is

$$\{\{C[s_1]_p\}, 1, w_{\mathbf{can}}(C[s_1]_p), l_1, r_1, (\{C[t_1]_p\}, 0, -, -, -)^{\{0,1\}}\}$$

and the measure of  $C[\mathcal{P}_2]_p$  is  $(\{C[s_2]_p\}, 0, -, -, -)$ , where  $t_1 \prec s_1 \prec s_2$ .  $C[\mathcal{P}_1]_p$  is strictly smaller than  $C[\mathcal{P}_2]_p$ .



- if  $\mathcal{P}_1$  is a  $\rightarrow_{R_{\text{can}}}$ -step, and  $\mathcal{P}_2$  is a  $\rightsquigarrow$  step, necessarily, the first component weakly decreases and the second component strictly decreases.  
The measure of  $C[\mathcal{P}_1]_p$  is  $(\{\{C[s_1]_p\}\}, 0, -, -, -)$  which is strictly smaller than the measure of  $C[s_2]_p \rightsquigarrow_{l_2 \rightarrow r_2} C[t_2]_p$ , that is  $(\{\{C[s_2]_p\}\}, 1, w_{\text{can}}(C[s_2]_p), l_2, r_2)$  since  $s_1 \preceq s_2$ .
- if  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are  $\rightsquigarrow$ -steps, they are of the form

$$s_i \rightsquigarrow_{l_i \rightsquigarrow r_i} t_i$$

and the corresponding measures are  $(\{\{s_i\}\}, 1, w_{\text{can}}(s_i), l_i, r_i)$ . The measure of  $C[\mathcal{P}_1]_p$  is

$$\begin{aligned} & (\{\{C[s_1]_p\}\}, 1, w_{\text{can}}(C[s_1]_p), l_1, r_1), \\ & (\{\{C[t_1]_p\}\}, 0, w_{\text{can}}(C[t_1]_p), C[t_1]_p, \text{can}(C[t_1]_p))^{(0,1)} \} \end{aligned}$$

and the measure of  $C[s_2]_p \rightsquigarrow_{l_2 \rightarrow r_2} C[t_2]_p$  is  $(\{\{C[s_2]_p\}\}, 1, w_{\text{can}}(C[s_2]_p), l_2, r_2)$ .

If  $s_1 < s_2$ , since  $t_1 < s_1$ ,  $C[\mathcal{P}_1]_p$  is strictly smaller than  $C[s_2]_p \rightsquigarrow_{l_2 \rightarrow r_2} C[t_2]_p$ . Otherwise,  $s_1 \simeq s_2$  and  $s_1 =_{AC} s_2$ . Hence  $w_{\text{can}}(s_1) = w_{\text{can}}(s_2)$  and the decrease occurs on the last two components. Therefore  $(\{\{C[s_1]_p\}\}, 1, w_{\text{can}}(C[s_1]_p), l_1, r_1)$  and  $(\{\{C[t_1]_p\}\}, 0, w_{\text{can}}(C[t_1]_p), C[t_1]_p, \text{can}(C[t_1]_p))$  are strictly smaller than  $(\{\{C[s_2]_p\}\}, 1, w_{\text{can}}(C[s_2]_p), l_2, r_2)$ .

- When a step is an equational step, necessarily the decrease occurs on the first component. Since  $<$  is compatible with plugging terms under a context, hence the wanted result.

We can now come to the proof of Lemma 9.

*Proof* Let us denote by  $l \rightarrow r$  the rule of  $R_\omega$ , and  $g \rightarrow d$  the rule of  $R_{\text{can}}$ ; since  $l$  is in a canonical form (invariant of the completion run), the reduction using  $g \rightarrow d$  can only take place at a position  $q$  which is above or parallel to the position  $p$  of the reduction using  $l \rightarrow r$ . We prove by induction that there exists a proof between  $s$  and  $s'$  which is strictly smaller than the original peak.

- In the parallel case, the subproof

$$s \begin{array}{c} \xleftarrow{p} \rightsquigarrow t \xrightarrow{q} s' \\ \xleftarrow{r} \xleftarrow{l} \quad \xrightarrow{g} \xrightarrow{d} \end{array}$$

can be seen as

$$\text{can}(t[r]_p[g]_q) \xleftarrow{R_{\text{can}}} t[r]_p[g]_q \xleftarrow{r \leftarrow l} t[l]_p[g]_q \xrightarrow{R_{\text{can}}} t[l]_p[d]_q$$

Notice that  $t[r]_p[g]_q$  and  $t[r]_p[d]_q$  are equal modulo AC, X, hence have the same canonical form. The above subproof can be replaced by

$$\text{can}(t[r]_p[g]_q) = \text{can}(t[r]_p[d]_q) \xleftarrow{R_{\text{can}}} t[r]_p[d]_q \xleftarrow{r \leftarrow l} t[l]_p[d]_q$$

which is actually

$$s \begin{array}{c} \xleftarrow{p} \rightsquigarrow s' \\ \xleftarrow{r} \xleftarrow{l} \end{array}$$

The measure strictly decreases, since for the first subproof it is equal to  $(\{\{t\}\}, 1, 1, l, r)$ ,  $(\{\{t\}\}, -, -, -, -)$ , and for the second one, it is equal to  $(\{\{s'\}\}, 1, w_{\text{can}}(s'), l, r)$ , and  $s' \preceq t$ .

- In the prefix case, we first prove the wanted result when the position  $q$  is equal to  $\Lambda$ . Now we make an induction over  $p$ , in order to establish that there is a proof between  $s$  and  $s'$ , with a measure (weakly) smaller than  $s \stackrel{\leftarrow}{\sim}_{r \leftarrow l} t$ , hence strictly smaller than the global measure of the peak. If  $p = \Lambda$ , rewriting at top with a rule of  $R_\omega$  is impossible if it is not an extended rewriting, since  $l$  is in a canonical form. In the extended case, the subproof to be replaced has the form

$$\text{can}(u(r, l')) \stackrel{\leftarrow}{\sim}_{r \leftarrow l} t \xrightarrow{R_{\text{can}}} s'$$

where  $t =_{AC} u(l, l')$ , and  $s' = \text{can}(u(l, l'))$ . By definition of  $\text{can}$  and since  $l$  is in a canonical form and  $u$  is an AC symbol,  $s'$  is AC-equal to  $u(l, \text{can}(l'))$ . The subproof can be replaced by

$$\text{can}(u(r, l')) = \text{can}(u(r, \text{can}(l'))) \stackrel{\leftarrow}{\sim}_{r \leftarrow l} u(l, \text{can}(l')) =_{AC} s'$$

where the identity  $\text{can}(u(r, \text{can}(l'))) = \text{can}(u(r, l'))$  holds since  $u(r, \text{can}(l'))$  and  $u(r, l')$  are equal modulo AC, X. The measure strictly decreases, since for the first subproof it is equal to  $\{\{\{t\}, 1, w_{\text{can}}(t), l, r), (\{t\}, \rightarrow, \rightarrow, -)\}$ , and for the second one, it is equal to  $\{\{\{s'\}, 1, w_{\text{can}}(s'), l, r)\}$ , and  $s' \prec t$ , or  $s' \simeq t$  with  $w_{\text{can}}(s') = w_{\text{can}}(t)$ .

If  $p$  is of the form  $i \cdot p'$ ,  $t$  is of the form  $f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n)$ , and the proof to be replaced

$$\text{can}(f(t_1, \dots, t_i[r]_{p'}, \dots, t_n)) \stackrel{\leftarrow}{\sim}_{r \leftarrow l} f(t_1, \dots, t_i[l]_{p'}, \dots, t_n) \xrightarrow{R_{\text{can}}} s'$$

We may assume without loss of generality that  $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$  are in a canonical form, since

$$\begin{aligned} s' = \text{can}(t) &= \\ & \text{can}(f(\text{can}(t_1), \dots, \text{can}(t_{i-1}), t_i[l]_{p'}, \text{can}(t_{i+1}), \dots, \text{can}(t_n))) \\ \text{can}(f(t_1, \dots, t_i[r]_{p'}, \dots, t_n)) &= \\ & \text{can}(f(\text{can}(t_1), \dots, \text{can}(t_{i-1}), t_i[r]_{p'}, \text{can}(t_{i+1}), \dots, \text{can}(t_n))) \end{aligned}$$

We also denote as

$$s_0 = f(t_1, \dots, \text{can}(t_i[r]_{p'}), \dots, t_n)$$

and

$$s'_0 = f(t_1, \dots, \text{can}(t_i[l]_{p'}), \dots, t_n)$$

We know that  $\text{can}(t_i[l]_{p'}) \preceq t_i[l]_{p'}$ , and we distinguish between two cases.

- If  $\text{can}(t_i[l]_{p'}) \prec t_i[l]_{p'}$ , then by induction hypothesis, there exists a proof  $\mathcal{P}$  between  $\text{can}(t_i[r]_{p'})$  and  $\text{can}(t_i[l]_{p'})$  which is weakly smaller than

$$\text{can}(t_i[r]_{p'}) \stackrel{\leftarrow}{\sim}_{r \leftarrow l} t_i[l]_{p'}$$

The decreasing is actually strict since an equivalent proof should be in one step, and the only possibility is a step of the form

$$\text{can}(t_i[r]_{p'}) \stackrel{\leftarrow}{\sim}_{r \leftarrow l} \text{can}(t_i[l]_{p'})$$

However since  $\text{can}(t_i[l]_{p'}) \prec t_i[l]_{p'}$  and  $w_{\text{can}}(t_i[l]_{p'}) = w_{\text{can}}(t_i[l]_{p'})$  cannot be not simultaneously true, such an equivalent step is not possible. Among all possible proofs  $\mathcal{P}$ , we pick up a minimal one. By the previous lemmas,  $\mathcal{P}$  does not contains  $\rightarrow_{R_\infty}$  steps, hence  $f(t_1, \dots, \mathcal{P}, \dots, t_n)$  is strictly smaller than

$$\text{can}(s_0) \xleftrightarrow[r \leftarrow l]{\leftarrow} t$$

If we consider the proof  $\mathcal{P}'$

$$s \xleftrightarrow[R_{\text{can}}]{\{0,1\}} s_0 \xleftarrow{f(t_1, \dots, \mathcal{P}, \dots, t_n)} s'_0 \xrightarrow[R_{\text{can}}]{\{0,1\}} s'$$

all its elementary steps are strictly smaller than  $(\{\{t\}\}, 1, 1, l, r)$ . We have seen that this is true for the middle part, and also for the left part  $(\{\{s_0\}\}, 0, 1, s_0, s)^{\{0,1\}}$ , and the right part  $(\{\{s'_0\}\}, 0, 1, s'_0, s')^{\{0,1\}}$ .

$\mathcal{P}'$  is a proof between  $s$  and  $s'$  which is strictly smaller than  $s \xleftrightarrow[r \leftarrow l]{\leftarrow} t$ .

- If  $\text{can}(t_i[l]_{p'}) \simeq t_i[l]_{p'}$ , then by the AC-totality of  $\preceq$ ,  $\text{can}(t_i[l]_{p'}) =_{\text{AC}} t_i[l]_{p'}$ . Since  $s' = \text{can}(t)$ , we know that  $s' \preceq t$  and we make a case analysis:

- If  $s' \simeq t$  then  $s'$  is actually  $\text{can}_{\text{AC}}(t)$  which is AC-equal to  $t$ .  $s'$  contains  $t_i[l]_{p'}$  as a subterm and can be reduced with  $l \rightarrow r$  to  $\text{can}(s'[t_i[r]_{p'}])$  which is AC-equal to  $t[t_i[r]_{p'}]_i$ . Hence  $\text{can}(s'[t_i[r]_{p'}]) = \text{can}(t[t_i[r]_{p'}]_i) = s$  and the proof

$$s \xleftrightarrow[r \leftarrow l]{\leftarrow} s'$$

is equivalent to, hence weakly smaller than  $s \xleftrightarrow[r \leftarrow l]{\leftarrow} t$ .

- If  $s' \prec t$ , then we can first see the peak as follows:

$$s \xleftarrow[R_{\text{can}}]{\{0,1\}} s_0 \xleftarrow[r \leftarrow l]{\leftarrow} t \rightarrow_{R_{\text{can}}} s' = \text{can}(t)$$

We eagerly replace every occurrence of  $l$  by  $r$  in  $s_0$  and  $s'$ , getting respectively  $s_1$  and  $s''$ . Then  $s_1$  and  $s''$  are equal modulo AC and X, because any proof modulo AC and X between  $t$  and  $s'$  can be replayed by replacing the  $\sigma$ -instances of AC and X used originally by  $\sigma'$ -instances where  $x\sigma'$  is  $x\sigma$  where every occurrence of  $l$  is replaced by  $r$ . We get the new proof

$$s \xleftarrow[R_{\text{can}}]{\{0,1\}} s_0 \xrightarrow[l \rightarrow r]{*} s_1 \xrightarrow[R_{\text{can}}]{\{0,1\}} \text{can}(s_1) = \text{can}(s'') \xleftarrow[R_{\text{can}}]{\{0,1\}} s' = \text{can}(t)$$

Since  $s' \prec t$ , all terms in the above proof are strictly smaller than  $t$ , hence the measure of this proof is strictly smaller than  $(\{\{t\}\}, 1, 1, l, r)$ .

If the proof occurs under a context  $t[\bullet]_q$ , we know that there is a proof  $\mathcal{P}$  between  $s = \text{can}(t[r]_{q-p'})$  and  $\text{can}(t)$  which is weakly smaller than  $(\{\{t[l]_{q-p'}\}\}, 1, 1, l, r)$  (case  $\rightarrow_{R_{\text{can}}}$  at  $\Lambda$ ). Hence

$$s \xleftrightarrow[\mathcal{P}]{\leftarrow} \text{can}(t) \xleftarrow[R_{\text{can}}]{\{0,1\}} s'$$

is a proof between  $s$  and  $s'$  which is weakly smaller than

$$\{\{(\{\{t[l]_{q-p'}\}\}, 1, 1, l, r), (\{\{s'\}\}, 0, 1, s', \text{can}(t))^{\{0,1\}}\}\}$$

whereas the measure of the original peak is

$$\{\{\{t\}\}, 1, 1, l, r\}, (\{\{t\}\}, 0, 2, t, s')\}$$

Since  $s' \preceq t$ , the measure of the new proof is strictly smaller than the measure of the original peak.

**Theorem 3** *If  $s$  and  $t$  are two terms such that  $s \xleftrightarrow{*}_{AC, X, E_\infty, R_\infty} s'$  then  $\mathbf{can}(s) \downarrow_{R_\omega} = \mathbf{can}(t) \downarrow_{R_\omega}$ .*

*Proof* If  $s$  and  $s'$  are equal modulo  $\xleftrightarrow{*}_{AC, X, E_\infty, R_\infty}$ , so are  $\mathbf{can}(s)$  and  $\mathbf{can}(s')$ . By the above lemmas, a minimal proof between  $\mathbf{can}(s)$  and  $\mathbf{can}(s')$  is necessary of the form  $\mathbf{can}(s) (\rightsquigarrow_{R_\omega} \cup \rightarrow_{R_{\mathbf{can}}})^* (\leftarrow_{R_\omega} \cup \leftarrow_{R_{\mathbf{can}}})^* \mathbf{can}(s')$ . This sequence of steps can also be seen as  $\mathbf{can}(s) \rightarrow_{R_{\mathbf{can}}}^* (\rightsquigarrow_{R_\omega} \rightarrow_{R_{\mathbf{can}}}^*)^* (\leftarrow_{R_{\mathbf{can}}}^* \leftarrow_{R_\omega})^* \leftarrow_{R_{\mathbf{can}}}^* \mathbf{can}(s')$ . By definition  $\rightarrow_{R_{\mathbf{can}}}$  cannot follow a  $\rightsquigarrow_{R_\omega}$ -step, and  $\mathbf{can}(s)$  and  $\mathbf{can}(s')$  cannot be reduced by  $\rightarrow_{R_{\mathbf{can}}}$ , hence the wanted result.

### 5.3 Termination

The proof of termination partly reuses some facts used for the termination proof of AC-ground completion (based on Higman's lemma), but also needs some intermediate lemmas which are specific to our framework<sup>6</sup>. We shall prove that, under a strongly fair strategy,  $R_\omega$  is finite and obtained in a finite time (by cases on the head function symbol of the rule's left-hand side), and then we show that  $R_\omega$  will clean up the next configurations and the completion process eventually halts on  $\langle \emptyset \mid R_\omega \rangle$ . In order to make our case analysis on rules, and to prove the needed invariants, we define several sets of terms (assuming without loss of generality that  $E_0 = \mathbf{can}(E_0)$ ):

$$\begin{aligned} T_0 &= \{t \mid \exists t_0, e_1, e_2 \in \mathcal{T}_\Sigma(\mathcal{X}), e_1 \approx e_2 \in E_0 \text{ and } t_0 = e_i|_p \text{ and } t_0 \rightsquigarrow_{R_\infty}^* t\} \\ T_{0X} &= T_0 \cup \{f_X(t_1, \dots, t_n) \mid f_X \in \Sigma_X \text{ and } \forall i, t_i \in T_{0X}\} \\ T_1 &= \{t \mid t \in T_0 \text{ and } \forall p, t|_p \in T_{0X}\} \\ T_2 &= \{u(t_1, \dots, t_n) \mid 2 \leq n \text{ and } u \in \Sigma_{AC} \text{ and } \forall i, t_i \in T_1\} \end{aligned}$$

$T_0$  is the set of all terms and subterms in the original problem as well as their reducts by  $R_\infty$ . The set  $T_{0X}$  moreover contains terms with X-aliens in  $T_0$ .  $T_1$  is the set of terms that can be introduced by X from terms of  $T_0$  (by solving or canonizing).  $T_2$  is a superset of the terms built by critical pairs.

**Lemma 11**  $\forall \gamma, t, s, \gamma \in R_\infty \cap T_j^2 \wedge t \in T_i \wedge t \rightsquigarrow_\gamma s \implies s \in T_i$ , for  $i, j = 1, 2$ .

The proof is by structural induction over terms (for dealing with rewriting under a context) and by case analysis over  $T_i$  when rewriting at the top level. It uses the (quasi-immediate) fact that  $T_0 \cap T_2 \subseteq T_1$ .

**Lemma 12** *For all accessible configuration  $\langle E_n \mid R_n \rangle$ ,  $E_n \cup R_n \subseteq T_1^2 \cup T_2^2$ .*

<sup>6</sup> We assume that  $\perp$  is not encountered, otherwise, termination is obvious.

The proof is by induction over  $n$ , and uses Lemma 11.

The first step of the termination proof is to show that  $R_\omega \cap T_1^2$  is finite (Lemma 14). It is specific to our framework, due to the presence of  $X^7$ .

**Lemma 13** *Under a strongly fair strategy, if  $l \rightarrow r_n$  is created at step  $n$  in  $R_n$  and  $l \rightarrow r_m$  at step  $m$  in  $R_m$ , with  $n < m$ , then  $r_m$  is a reduct of  $r_n$  by  $\rightsquigarrow_{R_\infty}$ .*

*Proof* The proof is by induction over the length of the derivation, and by case analysis over the rule which has been applied.

- **Orient** applied on  $s = t$  cannot create a new rule  $p \rightarrow v$  with an already present left hand side, because the strongly fair strategy implies that  $s$  and  $t$  are fully reduced, and the new left hand side  $p$  is a subterm of  $s$  or  $t$ .
- **Simplify**, **Collapse** and **Deduce** do not create a new rule.
- **Compose** obviously preserves the invariant.

**Corollary 1** *Under a strongly fair strategy,  $R_\infty$  is finitely branching.*

*Proof* If  $R_\infty$  is not finitely branching, there exist an infinite sequence of rules  $(l \rightarrow r_n)_n$  where  $l \rightarrow r_n$  first appears in  $\langle E_n \mid R_n \rangle$ . Thanks to Lemma 13, since  $R_\infty$  is included in  $\prec$ , the sequence  $(r_n)_n$  is strictly decreasing w.r.t  $\prec$ . The well-foundedness of  $\prec$  contradicts the infinity of  $(r_n)_n$ .

**Lemma 14** *Under a strongly fair strategy, the set of rules in  $R_\omega \cap T_1^2$  is finite.*

*Proof* If  $l \rightarrow r$  belongs to the set  $R_\omega \cap T_1^2$ ,  $l$  is reduct of a term  $l_0$  in  $E_0$  by  $\rightsquigarrow_{R_\infty}$ . Since  $\rightsquigarrow_{R_\infty}$  is terminating (because it is included in  $\prec$ ), and finitely branching (above corollary), any term has finitely many reducts by  $\rightsquigarrow_{R_\infty}$ . In particular since  $E_0$  is finite, there are finitely many possible left-hand side. Moreover since in  $R_\omega$  two distinct rules have distinct left-hand sides,  $R_\omega \cap T_1^2$  is finite.

Here is the second step of the termination proof, finiteness of  $R_\omega \cap T_2^2$ , which is mostly the same as in the usual AC-ground completion:

**Lemma 15** *The set of persistent rules in  $R_\omega$  which are in  $T_2^2$  is finite.*

*Proof* The set  $R_\omega \cap T_2^2$  can be divided into a finite union of sets, according to the top AC function symbol of the left hand-side of the rules. We shall prove that for each  $u \in \Sigma_{AC}$ , the corresponding subset is finite.

Let  $u$  be a fixed AC function symbol, and let  $u(l_1, \dots, l_n) \rightarrow r$  be a rule of  $R_\omega \cap T_2^2$ . By definition of  $T_2$ , and by the soundness of  $R_\infty$ , each  $l_i$  is equal modulo  $ACX, E_0$  to a term  $l_i^0$  in  $E_0$ . Since  $l_i$  is irreducible by  $R_\omega$  (otherwise the rule  $u(l_1, \dots, l_n) \rightarrow r$  would have collapsed), there is a rewriting proof  $l_i \rightsquigarrow_{R_\omega}^* l_i^0$ . Notice that two distinct rules in  $R_\omega$  have some distinct left-hand sides (otherwise one would have collapsed the other) (this implies in particular that  $R_\omega$  is finitely branching). Since  $\rightsquigarrow_{R_\omega}$  is included in a well-founded ordering, and is finitely branching any term has a finite number of reducts. Since  $E_0$  is finite, each  $l_i$  belongs to the *finite* set of reducts  $Red(E_0)$  of  $E_0$  by  $\rightsquigarrow_{R_\omega}$ . By Higman's lemma, if there is an *infinite* number of rules where the left-hand side is of the form  $u(t_1, \dots, t_n)$ , there

<sup>7</sup>  $X$  may change the head function symbol of terms in an equational proof, which is not the case of AC in standard ground AC-completion.

exist two rules  $l \rightarrow r$  and  $l' \rightarrow r'$ , such that the multiset of arguments  $\{\{l_1, \dots, l_n\}$  of  $l$  is included in the multiset of arguments  $\{\{l'_1, \dots, l'_m\}$  of  $l'$ . This would imply that the second rule collapses by the first one, which contradicts its persistence. Hence the wanted result.

When  $R_\omega$  has been proven to be finite, we show that once it is obtained,  $R_\omega$  will “clean up” the configuration within a finite number of steps, hence the termination:

**Theorem 4** *Under a strongly fair strategy,  $AC(X)$  terminates.*

*Proof* When the strategy is strongly fair,  $R_\omega$  is finite. Moreover each rule in  $R_\omega$  is obtained within a finite number of steps. Once all persistent rules are present in the rules of the configuration  $\langle E \mid R \rangle$ , the rule **Orient** always returns an empty set of rules. If the measure of a configuration is the triple made of the number of remaining critical pairs to generate, the multiset of terms in  $R$  (compared with  $\prec$ ), and the number of equations on  $E$ , it strictly decreases.

## 6 Term Abstraction and Multiset Ordering

In this section, we show that a simple preprocessing step allows us to use a partial multiset ordering instead of a full AC-compatible reduction ordering in the  $AC(X)$  algorithm. This optimization is motivated by the fact that although AC-RPO orderings are suitable when proving termination of completion procedures, they are not easily implementable in practice. Our preprocessing step is similar to the **Extension** inference rule found in Abstract Congruence Closure [?].

Let  $K$  be a set of constant symbols disjoint from  $\Sigma$  and  $\mathcal{X}$  and  $\prec_{\mathcal{X}}$  be a total rewrite ordering on  $\mathcal{T}(\Sigma_{\mathcal{X}} \cup K)$ . We define two sets of terms  $\mathcal{T}_\emptyset$  and  $\mathcal{T}_{AC}$  as follows:

$$\mathcal{T}_\emptyset = \left\{ f(v_1, \dots, v_n) \left| \begin{array}{l} f \in \Sigma_\emptyset \quad \wedge \\ \text{arity}(f) = n \quad \wedge \\ \bigwedge_{i=1}^n v_i \in \mathcal{T}(\Sigma_{\mathcal{X}} \cup K) \end{array} \right. \right\}$$

$$\mathcal{T}_{AC} = \left\{ u(v_1, u(v_2, \dots, u(v_{n-1}, v_n) \dots)) \left| \begin{array}{l} u \in \Sigma_{AC} \quad \wedge \\ n \geq 2 \quad \wedge \\ \bigwedge_{i=1}^n v_i \in \mathcal{T}(\Sigma_{\mathcal{X}} \cup K) \end{array} \right. \right\}$$

In order to enable the use of a multiset ordering as an input for  $AC(X)$ , we have to transform the original set of ground equations  $E$  to a simpler one containing only *abstracted* equations.

**Definition 4 (Abstracted equations)** An equation  $s \approx t$  is said to be abstracted if one of the following statements holds:

1.  $s, t \in \mathcal{T}(\Sigma_{\mathcal{X}} \cup K)$
2.  $s \in (\mathcal{T}_\emptyset \cup \mathcal{T}_{AC})$  and  $t \in \mathcal{T}(\Sigma_{\mathcal{X}} \cup K)$
3.  $s, t \in \mathcal{T}_{AC}$  and  $s(A) = t(A)$

The set of all abstracted equations is denoted by  $\mathcal{A}$ .

Let  $\pi$  be an abstraction function from  $\mathcal{T}_{AC} \cup \mathcal{T}_\emptyset$  to  $K$  such that if  $\pi(s) = \pi(t)$  then  $s =_{AC, X} t$ . Given a set  $E^0$  of ground equations, the term abstraction of  $E^0$  consists in applying, as long as possible, the following inference rules starting from the initial configuration  $\langle E^0 \mid \emptyset \rangle$ .

$$\mathbf{Abstract1} \frac{\langle E \uplus \{s \approx t\} \mid E_{\mathcal{A}} \rangle}{\langle E \mid E_{\mathcal{A}} \cup \{s \approx t\} \rangle} s \approx t \in \mathcal{A}$$

$$\mathbf{Abstract2} \frac{\langle E \cup \mathcal{C}[f(\mathbf{v})] \approx t \mid E_{\mathcal{A}} \rangle}{\langle E \cup \mathcal{C}[k] \approx t \mid E_{\mathcal{A}} \cup \{f(\mathbf{v}) \approx k\} \rangle} \mathcal{C}[f(\mathbf{v})] \approx t \notin \mathcal{A}$$

where,

1.  $f(\mathbf{v}) \in (\mathcal{T}_\emptyset \cup \mathcal{T}_{AC})$
2.  $k = \pi(f(\mathbf{v}))$

The propositions 1 and 2 state respectively the termination and the correctness of the abstraction process.

**Proposition 1** *The application of the rules **Abstract1** and **Abstract2** terminates and produces a configuration of the form  $\langle \emptyset \mid E_{\mathcal{A}}^\infty \rangle$ , where  $E_{\mathcal{A}}^\infty \subseteq \mathcal{A}$ .*

*Proof* The proof of termination is immediate using a decreasing measure. The size of a configuration is equal to the total sum of the sizes of the terms in its first component. Here, the size of a term is recursively defined in a standard way with 1 for the size of constants in  $K$ , and 2 for the size of other constants.

It remains to show that if a configuration is of the form  $\langle E \mid E_{\mathcal{A}} \rangle$  and  $E \neq \emptyset$ , at least one rule applies. Let  $s \approx t$  be an equation in  $E$ . If  $s \approx t \in \mathcal{A}$  the **Abstract1** applies. Otherwise, since  $s \approx t \notin \mathcal{A}$ , by condition 1. of Definition 4, there is a minimal subterm of  $s$  or  $t$  which does not belong to  $\mathcal{T}(\Sigma_X \cup K)$ . This term thus has a suitable form to fulfill condition 1. in the rule **Abstract2** which applies.

**Proposition 2** *Let  $\langle E^0 \mid \emptyset \rangle \rightarrow^* \langle \emptyset \mid E_{\mathcal{A}}^\infty \rangle$  be a fixed run of the abstraction process. For any terms  $s, t \in \mathcal{T}(\Sigma, \emptyset)$ , we have:*

$$s =_{E^0, X, AC} t \iff s =_{E_{\mathcal{A}}^\infty, X, AC} t$$

*Proof* The direction  $\Rightarrow$  is immediate for **Abstract1**. For **Abstract2**, it rests on the fact that a step using  $\mathcal{C}[f(\mathbf{v})] \approx t$  can be replaced by two steps, the first one using  $f(\mathbf{v}) \approx k$  and the second one using  $\mathcal{C}[k] \approx t$ .

In order to prove  $\Leftarrow$ , we use the following invariant: if  $\langle E \mid E_{\mathcal{A}} \rangle \rightarrow \langle E' \mid E'_{\mathcal{A}} \rangle$ ,  $s =_{E', E'_{\mathcal{A}}, X, AC} t$  and  $s$  and  $t$  do not contain any constant in  $K$ , then  $s =_{E, E_{\mathcal{A}}, X, AC} t$ . This is immediate when the rule **Abstract1** is applied. When **Abstract2** replaces  $\mathcal{C}[f(\mathbf{v})] \approx t$  by  $\{f(\mathbf{v}) \approx k, \mathcal{C}[k] \approx t\}$ , we first replace every step using  $\mathcal{C}[k] \approx t$  by a compound step using  $\mathcal{C}[k] \approx \mathcal{C}[f(\mathbf{v})]$  followed by  $\mathcal{C}[f(\mathbf{v})] \approx t$ . Then all occurrences of  $k$  are replaced by  $f(\mathbf{v})$  in intermediate terms, and the now useless steps using  $f(\mathbf{v}) \approx f(\mathbf{v})$  (former  $f(\mathbf{v}) \approx k$ ) are removed. The transformed proof is now in  $=_{E, E_{\mathcal{A}}, X, AC}$ , and since neither  $s$  nor  $t$  contain constants in  $K$ , they are not affected by these transformations.

Now that we have shown how to abstract the initial set of equations  $E$ , we will define the reduction ordering  $\prec$  that we will use in  $\text{AC}(\mathbf{X})$ . We do not need this ordering to be total on the terms in  $\mathcal{T}(\Sigma_{\mathbf{X}} \cup K, \emptyset) \cup \mathcal{T}_{\emptyset} \cup \mathcal{T}_{AC}$ . We only need a partial reduction ordering which allows us to get well oriented rewriting rules from the abstracted equations. Let  $\prec_{\mathbf{X}}^{mset}$  be the multiset extension of  $\prec_{\mathbf{X}}$ . Our reduction ordering is defined by:

1.  $\forall v_1, v_2 \in \mathcal{T}(\Sigma_{\mathbf{X}} \cup K), v_1 \prec_{\mathbf{X}} v_2 \implies v_1 \prec v_2$
2.  $\mathcal{T}(\Sigma_{\mathbf{X}} \cup K) \prec \mathcal{T}_{\emptyset}$
3.  $\mathcal{T}(\Sigma_{\mathbf{X}} \cup K) \prec \mathcal{T}_{AC}$
4.  $\forall u(\mathbf{v}_1), u(\mathbf{v}_2) \in \mathcal{T}_{AC}, \{\{\mathbf{v}_1\}\} \prec_{\mathbf{X}}^{mset} \{\{\mathbf{v}_2\}\} \implies u(\mathbf{v}_1) \prec u(\mathbf{v}_2)$

After that, we have to show that  $\text{AC}(\mathbf{X})$  does not introduce non-abstracted equations when collapsing rules, computing critical pairs, using canonized rewriting, and solving equations. Hence, the following lemma:

**Lemma 16** *For any configuration  $\langle E_n^\infty \mid R_n \rangle$  reachable from  $\langle E_{\mathcal{A}}^\infty \mid \emptyset \rangle$ , we have:*

$$\forall (s, t) \in (E_n^\infty \cup R_n), \quad s \approx t \in \mathcal{A}$$

*Proof* The lemma obviously holds for the initial state. For the induction step, we can easily show that the abstracted form of equations is preserved by canonized rewriting wrt an abstracted rule, hence so as when applying the inference rules **Simplify**, **Compose** and **Collapse**. Concerning **Deduce**, we notice by inspecting the definition of **headCP**, that when  $l \rightarrow r$  and  $l' \rightarrow r'$  are abstracted oriented equations, so is the resulting critical pair. The only subtle case is **Orient**, in particular when solving an equation  $s \approx t$ , with  $s \in \mathcal{T}(\Sigma_{\mathbf{X}} \cup K)$  and  $t \in \mathcal{T}_{\emptyset} \cup \mathcal{T}_{AC}$ . Due to the definition of  $\prec$  and to the fact that the solver has to fulfill the ordering constraints stated in Axiom 1, the solution of  $s \approx t$  has to be  $t \mapsto s$ .

Finally, we notice that  $\prec$  is a suitable ordering for the  $\text{AC}(\mathbf{X})$  completion procedure since on the equations in  $\mathcal{A}$ , it coincides with the AC-RPO ordering based on a precedence  $\prec_p$  such that  $\Sigma_{\mathbf{X}} \prec_p K \prec_p \Sigma_{\mathcal{E}} \cup \Sigma_{AC}$ .

## 7 Experimental Results

We implemented the  $\text{AC}(\mathbf{X})$  algorithm as well as a preprocessing step that enables the use of a partial multiset reduction ordering (see Section 6). As described in Section 4, the state of the procedure is a pair  $\langle E \mid R \rangle$  of equations and rules. We apply the following strategy for processing an equality  $u \approx v \in E$ :

**Sim\*** (**Tri** | **Bot** | (**Ori** (**Com Col Ded**)\*))

First,  $u \approx v$  is simplified as much as possible by **Simplify**. Then, if it is not proven to be trivially solved by **Trivial** or unsolvable by **Bottom**, it is solved by **Orient**. Each resulting rule is added to  $R$  and then used to **Compose** and **Collapse** the other rules of  $R$ . Critical pairs are then computed by **Deduce**.

We benchmark  $\text{AC}(\mathbf{X})$  and compare its performances with our own SMT solver ALT-ERGO [?] and some state-of-the-art solvers (Z3 v2.8, CVC3 v2.2, SIMPLIFY



v1.5.4). All measures are obtained on a laptop running Linux equipped with a 2.58GHz dual-core Intel processor and with 4Gb main memory. Provers are given a time limit of five minutes for each test and memory limitation is managed by the system. The results are given in seconds; we write TO for *timeout* and OM for *out of memory*.

Our test suite is made of crafted *ground* formulas which are valid in the combination of the theory of linear arithmetic LA, the free theory of equality  $\mathcal{E}$  and a small part of the theory of sets defined by the symbols  $\cup$ ,  $\subseteq$ , the singleton constructor  $\{\cdot\}$ , and the following axioms:

$$\mathcal{S}_{\cup} \begin{cases} \text{Assoc} : \forall x, y, z. x \cup (y \cup z) \approx (x \cup y) \cup z \\ \text{Commut} : \forall x, y. x \cup y \approx y \cup x \end{cases}$$

$$\mathcal{S}_{\subseteq} \begin{cases} \text{SubTrans} : \forall x, y, z. x \subseteq y \wedge y \subseteq z \Rightarrow x \subseteq z \\ \text{SubSuper} : \forall x, y, z. x \subseteq y \Rightarrow x \subseteq y \cup z \\ \text{SubUnion} : \forall x, y, z. x \subseteq y \Rightarrow x \cup z \subseteq y \cup z \\ \text{SubRefl} : \forall x. x \subseteq x \end{cases}$$

The theories  $\mathcal{E}$  and LA are built-in for all SMT solvers we use for our experiments. However, contrarily to AC(X) which also natively handles associativity and commutativity, SMT solvers use a generic mechanism for instantiating the axioms  $\mathcal{S}_{\cup}$  to reason modulo the AC properties of  $\cup$ .

In order to get the most accurate information about AC(X), we first benchmark a stand-alone version of our algorithm on ground formulas that can be proved without  $\mathcal{S}_{\subseteq}$ . In a second step, we consider ground formulas that are only provable with some axioms in  $\mathcal{S}_{\subseteq}$ . Since these axioms are not directly handled by AC(X), we benchmark a modified version of ALT-ERGO (to benefit from its instantiation mechanism) with AC(X) as its core decision procedure.

In the following, we use the standard mathematical notation  $\bigcup_{i=1}^d a_i$  for the terms of the form  $a_1 \cup (a_2 \cup (\dots \cup a_d)) \dots$  and we write  $\bigcup_{i=1}^d a_i; b$  for terms of the form  $a_1 \cup (a_2 \cup (\dots \cup (a_d \cup b))) \dots$ .

### 7.1 Benchmark of a stand-alone AC(X)

We consider two categories of formulas. The first category  $C_1$  is of the form

$$\bigwedge_{p=1}^n (\{e\} \cup \bigcup_{i=1}^d a_i^p) \approx b^p \implies \underbrace{\bigwedge_{p=1}^{n-1} \bigwedge_{q=p+1}^n \bigcup_{i=d}^1 a_i^p; b^q \approx \bigcup_{i=d}^1 a_i^q; b^p}_G$$

and the second category  $C_2$  is of the form

$$\bigwedge_{p=1}^n (\{t_p - p\} \cup \bigcup_{i=1}^d a_i^p) \approx b^p \wedge \bigwedge_{p=1}^{n-1} t_p + 1 \approx t_{p+1} \implies G$$

Notice that  $n$  is the number of hypothesis equations and  $d$  is the maximal depth of AC terms.

Proving the validity of  $C_1$ -formulas only requires the theory  $\mathcal{E}$  and the AC properties of the union symbol. These formulas are directly provable by AC( $\emptyset$ ) and

the results for this instance are given in the first column of the table in Figure 5. In order to prove  $C_1$ -formulas with SMT solvers, the axioms in  $\mathcal{S}_\cup$  have to be put in their context. The last four columns of the table contain the results for ALT-ERGO, Z3, CVC3 and SIMPLIFY.

$n, d$	AC( $\emptyset$ )	ALT-ERGO	Z3	CVC3	SIMPLIFY
3, 3	<b>0.01</b>	0.19	0.22	0.40	0.18
3, 6	<b>0.01</b>	32.2	OM	132	OM
3, 12	<b>0.01</b>	TO	OM	OM	OM
6, 3	<b>0.01</b>	11.2	1.10	13.2	2.20
6, 6	<b>0.02</b>	TO	OM	OM	OM
6, 12	<b>0.02</b>	TO	OM	OM	OM
12, 3	<b>0.16</b>	TO	5.64	242	11.5
12, 6	<b>0.24</b>	TO	OM	OM	OM
12, 12	<b>0.44</b>	TO	OM	OM	OM

**Fig. 5** The results for category  $C_1$

In order to prove the validity of  $C_2$ -formulas, the theory  $\mathcal{E}$ , the AC properties of  $\cup$  and the theory of linear arithmetic LA are required. These ground formulas are directly provable by AC(LA) and the results are given in the first column of the table in Figure 6. Similarly to category  $C_1$ , the last four columns of the table contain the results for the SMT solvers we considered. Again, the axioms  $\mathcal{S}_\cup$  have to be provided in the context, whereas linear arithmetic is directly handled by the built-in decision procedures of these provers.

$n, d$	AC(LA)	ALT-ERGO	Z3	CVC3	SIMPLIFY
3, 3	<b>0.01</b>	1.10	0.03	0.11	0.19
3, 6	<b>0.01</b>	TO	3.67	4.21	OM
3, 12	<b>0.01</b>	TO	OM	OM	OM
6, 3	<b>0.02</b>	149	0.10	2.26	2.22
6, 6	<b>0.02</b>	TO	17.7	99.3	OM
6, 12	<b>0.04</b>	TO	OM	OM	OM
12, 3	<b>0.27</b>	TO	0.35	44.5	11.2
12, 6	<b>0.40</b>	TO	76.7	TO	OM
12, 12	<b>0.72</b>	TO	OM	OM	OM

**Fig. 6** The results for category  $C_2$

## 7.2 Benchmark of ALT-ERGO with AC(X)

We now analyze the performances of AC(X) when it is used as the core decision procedure of ALT-ERGO. For that, we consider a third category  $C_3$  of formulas of the form

$$\bigwedge_{p=1}^n \bigcup_{i=1}^d \{e_i^p\} \approx b^p \wedge \bigcup_{i=1}^d \{e + e_i^p\} \approx c^p \wedge e \approx 0 \implies \bigwedge_{p=1}^n c^p \subseteq (b^p \cup \{e_d^p\}) \cup \{e\}$$

Proving the validity of  $C_3$ -formulas requires the theory  $\mathcal{E}$ , the AC properties of  $\cup$ , the theory of linear arithmetic LA and additionally some axioms in  $\mathcal{S}_{\subseteq}$ . We thus *only* provide the axioms  $\mathcal{S}_{\subseteq}$  in the context of the modified version of ALT-ERGO, whereas *all* the axioms in  $\mathcal{S}_{\subseteq}$  and  $\mathcal{S}_{\cup}$  are given in the context of the other SMT solvers. The results of this category are given in Figure 7.

$n, d$	ALT-ERGO with AC(LA)	ALT-ERGO	Z3	CVC3	SIMPLIFY
3, 3	<b>0.02</b>	3.16	0.09	10.2	OM
3, 6	<b>0.04</b>	TO	60.6	OM	OM
3, 12	<b>0.12</b>	TO	OM	OM	OM
6, 3	<b>0.07</b>	188	0.18	179	OM
6, 6	<b>0.12</b>	TO	TO	OM	OM
6, 12	<b>0.66</b>	TO	OM	OM	OM
12, 3	<b>0.20</b>	TO	0.58	OM	OM
12, 6	<b>0.43</b>	TO	TO	OM	OM
12, 12	<b>1.90</b>	TO	OM	OM	OM

**Fig. 7** The results for category  $C_3$

### 7.3 Benchmarks analysis

The results in Figures 5 and 6 show that, contrary to the axiomatic approach, built-in AC reasoning is little sensitive to the depth  $d$  of terms: given a fixed number  $n$  of equations, the running time is proportional to  $d$ . However, we notice a slowdown when  $n$  increases. This is due to the fact that AC(X) has to process a quadratic number of critical pairs generated from the equations in the hypothesis. From Figure 7, we remark that ALT-ERGO with AC(X) performs better than the other provers. The main reason is that its instantiation mechanism is not spoiled by the huge number of intermediate terms the other provers generate when they instantiate the AC axioms.

## 8 Instantiation Issues

Although AC(X) is effective on ground formulas, its integration as the core decision procedure of ALT-ERGO suffers from a *bad interaction* between the built-in treatment of AC and the axiom instantiation mechanism of ALT-ERGO which is roughly done as follows:

- each axiom of the form  $\forall \bar{x}. \mathcal{F}(\bar{x})$  provided in the context comes with a pattern  $P$  (also called *trigger*) which consists of a set of subterms of  $\mathcal{F}$  that covers  $\bar{x}$ ;
- the solver maintains a set  $G$  of *known* terms extracted syntactically from the *ground* literals that occur during its proof search;
- $G$  is partitioned into a set of equivalence classes according to the ground equalities currently known by the solver;

- new *ground* formulas  $\mathcal{F}\sigma$  are generated by matching  $P$  against  $G$  modulo the equivalence classes.

Let us show how this mechanism is used to prove the following ground formula:

$$(F_1) \quad (e \approx d \cup a \wedge b \subseteq d \wedge c \approx a \cup d) \Rightarrow b \cup a \subseteq c$$

For that, we only need to use the *SubUnion* axiom (defined in Section 7):

$$\text{SubUnion} : \quad \forall x, y, z. x \subseteq y \Rightarrow x \cup z \subseteq y \cup z$$

Let us assume that the pattern for this axiom is the term  $x \cup z \subseteq y \cup z$ . This pattern is matched against the term  $b \cup a \subseteq c$  by looking for a substitution  $\sigma$  such that  $(x \cup z \subseteq y \cup z)\sigma = b \cup a \subseteq c$  modulo the set of equivalence classes  $\{\{e, d \cup a, a \cup d, c\}, \{a\}, \{b\}, \{d\}, \{b \cup a\}, \{b \subseteq d\}, \{b \cup a \subseteq c\}\}$ . Such a substitution exists and maps  $x$  to  $b$ ,  $z$  to  $a$  and  $y$  to  $d$  since the term  $c$  is in the same class as  $d \cup a$ . The proof of  $F_1$  follows from the ground instance  $b \subseteq d \Rightarrow b \cup a \subseteq d \cup a$  of *SubUnion*.

Let us now explain the limitation of the interaction between AC(X) and the instantiation mechanism. The hypothesis  $e \approx d \cup a$  is useless (from a logical point of view) to prove  $b \cup a \subseteq c$ . Hence, the following formula  $F_2$  is equivalent to  $F_1$ :

$$(F_2) \quad (b \subseteq d \wedge c \approx a \cup d) \Rightarrow b \cup a \subseteq c$$

However, the cooperation of ALT-ERGO and AC(X) fails to prove  $F_2$ . The reason is that, since the term  $d \cup a$  does not syntactically occur in  $F_2$ , the equivalence classes are just  $\{\{a \cup d, c\}, \{a\}, \{b\}, \{d\}, \{b \cup a\}, \{b \subseteq d\}, \{b \cup a \subseteq c\}\}$  and the matching algorithm fails to match  $x \cup z \subseteq y \cup z$  against  $b \cup a \subseteq c$ .

## 9 Conclusion

We have presented a new algorithm AC(X) which efficiently combines, in the ground case, the AC theory with a Shostak theory X and the free theory of equality. Our combination consists in a tight embedding of the canonizer and the solver for X in ground AC-completion. The integration of the canonizer relies on a new rewriting relation, reminiscent to normalized rewriting, which interleaves canonization and rewriting rules. We proved the soundness of AC(X) by reusing standard proof techniques. Completeness is established thanks to a proofs' reduction argument, and termination follows the lines of the proof of ground AC-completion where the finitely branching result is adapted to account for the theory X. We showed how a simple preprocessing step allows us to get rid of a full AC-compatible reduction ordering, and to simply use a partial multiset extension of a *non necessarily AC-compatible* ordering.

AC(X) has been implemented in the ALT-ERGO theorem prover. The first experiments are very promising and show that a built-in treatment of AC, in the combination of the free theory of equality and a Shostak theory, is more efficient than an axiomatic approach for reasoning modulo AC.

As illustrated in Section 8, the main concern for using AC(X) as a core decision procedure in ALT-ERGO is that it does not saturate equivalent classes of ground known terms modulo AC. A naive (and incomplete) solution to this issue would

consist in adding, for each known ground AC-term  $t$ , a few number of AC equivalent terms (for instance by bounding the length of the AC equational proof between them). We rather plan to investigate a more elaborate solution which would consist in extending the pattern-matching algorithm of ALT-ERGO to exploit both ground equalities and properties of AC symbols. We also plan to extend AC(X) to handle the AC theory with unit or idempotence. This will be a first step towards a decision procedure for a substantial part of the finite sets theory. Another future work is the extension of AC(X) with a user defined first order rewriting system. This could be achieved by applying our combination technique to normalized rewriting and normalized completion [?].

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